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Zero temperature correlation functions for the impenetrable fermion gas

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Abstract
We calculate the long time and distance asymptotics of the one-particle correlation functions in the model of impenetrable spin 1/2 fermions in 1 + 1 dimensions. We consider the spin disordered zero temperature regime, which occurs when the limit $T \to 0$ is taken at a positive chemical potential. The asymptotic expressions are found from the asymptotic solution of the matrix Riemann–Hilbert problem related to the determinant representation of the correlation functions.

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1. Introduction

Consider the one-dimensional gas of spin 1/2 fermions interacting via the δ-potential. The Hamiltonian of this model is

$$H = \int \mathrm{d}x \left[- \sum_{\alpha=\uparrow,\downarrow} \psi_\alpha^\dagger(x)\partial_x^2\psi_\alpha(x) + Un_\uparrow(x)n_\downarrow(x) - \mu n(x) \right]. \tag{1.1}$$

The fermionic fields ψ_α (α is a spin index, $\alpha = \uparrow, \downarrow$) satisfy canonical equal-time anticommutational relations

$$\psi_\alpha(x)\psi_\beta^\dagger(y) + \psi_\beta^\dagger(y)\psi_\alpha(x) = \delta_{\alpha\beta}\delta(x-y). \tag{1.2}$$

The operators $n_\alpha(x) = \psi_\alpha^\dagger(x)\psi_\alpha(x)$ are the density operators for spin-up (spin-down) fermions and $n(x) = n_\uparrow(x) + n_\downarrow(x)$ is the total fermion density operator. In this paper we will consider the infinite U limit of one-particle correlation functions

$$G_h(x,t) = \left\langle \psi_\uparrow^\dagger(x,t)\psi_\uparrow(0,0) \right\rangle \tag{1.3}$$

and

$$G_e(x,t) = \langle \psi_\uparrow(x,t)\psi_\uparrow^\dagger(0,0)\rangle \tag{1.4}$$

where the average $\langle\ \rangle$ is taken over the thermal ensemble at a given temperature T and chemical potential μ. We will calculate the large x and t asymptotics of the correlation functions (1.3) and (1.4) assuming that (a) the limit of infinite repulsion $U = \infty$ is taken at finite positive chemical potential μ, and (b) the zero temperature limit is taken afterwards. The ground state of the system is infinitely degenerate at infinite U. This degeneracy leads to a non-trivial (non-conformal) behaviour of the correlation functions (1.3) and (1.4) at zero temperature. The physics of this phenomenon was presented earlier [1], here we present a detailed derivation of these results.

Our starting point is the determinant representation for $G_h(x,t)$ and $G_e(x,t)$, recently derived by Izergin and Pronko [2]. We relate the objects entering this representation to the solution of the corresponding matrix Riemann–Hilbert problem (RH problem) [3]. We perform the asymptotic analysis of this RH problem making use of previous work [4, 5]. The large x asymptotics of the correlation functions at $t = 0$ are given in section 4.3. The asymptotics of the time-dependent correlation functions in the space-like region are given in section 6.3. The asymptotics in the time-like region for $x/t = \text{const} > 0$ are given in section 8.2. The large t asymptotics of the correlation functions for $x = 0$ are given in section 9.

2. Determinant representation and the Riemann–Hilbert problem for the correlation functions of the model

This section is introductory. In section 2.1 we recall some elementary relations for the correlation functions (1.3) and (1.4). In section 2.2 we recall the results of paper [2] on the determinant representation for these correlation functions. In section 2.3 we examine the analytic properties of the functions entering this determinant representation. In section 2.5, following [3], we formulate the matrix RH problem related to the determinant representation of the correlation functions (1.3) and (1.4).

2.1. Elementary relations

In this subsection we recall some elementary relations for the correlation functions (1.3) and (1.4).

Due to the parity and the time reversal symmetry of the Hamiltonian (1.1) the functions (1.3) and (1.4) obey the following symmetry properties:

$$G_{h(e)}(x,t) = G_{h(e)}(-x,t) \tag{2.1}$$

and

$$G_{h(e)}(x,t) = G^*_{h(e)}(x,-t). \tag{2.2}$$

Here the asterisk stands for complex conjugation. Furthermore, the following relation between G_h and G_e can be written

$$G_h(x,t) = -G_e^*(x,t) + \langle \{\psi_\uparrow^\dagger(x,t), \psi_\uparrow(0,0)\}\rangle \tag{2.3}$$

where $\{\ ,\ \}$ stands for the anticommutator. At $t = 0$ this relation becomes

$$G_h(x,0) = -G_e^*(x,0) + \delta(x) \tag{2.4}$$

which is obvious from equation (1.2). Note also the scaling relation

$$G_{e(h)}(x, t; \mu) = \sqrt{\mu} G_{e(h)}(\sqrt{\mu} x, \mu t; 1) \tag{2.5}$$

where $G_{e(h)}(x, t; \mu)$ is the correlation function taken at a given chemical potential μ.

In the following we will assume that $x \geqslant 0, t \geqslant 0$ and $\mu = 1$. The correlation functions for all other cases will follow from relations (2.1), (2.2) and (2.5).

2.2. *Determinant representation for G_h and G_e*

In this subsection, following [2], we write down the determinant representation for the correlation functions (1.3) and (1.4). We use the notation of [3].

In [2] the following representation of the correlation functions (1.3) and (1.4) was derived

$$G_h(x, t) = \frac{\mathrm{e}^{-\mathrm{i}t}}{8\pi} \int_{-\pi}^{\pi} \mathrm{d}\eta \, F(\eta) B_{--}(x, t, \eta) \det(\hat{I} + \hat{V})(x, t, \eta) \tag{2.6}$$

and

$$G_e(x, t) = -\frac{\mathrm{e}^{\mathrm{i}t}}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\eta \, \frac{F(\eta)}{1 - \cos\eta} b_{++}(x, t, \eta) \det(\hat{I} + \hat{V})(x, t, \eta). \tag{2.7}$$

To shorten notation we will henceforth omit x, t and η in the arguments of all functions unless they are necessary.

The objects entering the representations (2.6) and (2.7) are defined as follows. The function $F(\eta)$ is

$$F(\eta) = 1 + \frac{\mathrm{e}^{\mathrm{i}\eta}}{2 - \mathrm{e}^{\mathrm{i}\eta}} + \frac{\mathrm{e}^{-\mathrm{i}\eta}}{2 - \mathrm{e}^{-\mathrm{i}\eta}}. \tag{2.8}$$

The determinant

$$\det(\hat{I} + \hat{V}) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{-1}^{1} \mathrm{d}k_1 \ldots \int_{-1}^{1} \mathrm{d}k_N \begin{vmatrix} V(k_1, k_1) & \cdots & V(k_1, k_N) \\ \vdots & \ddots & \vdots \\ V(k_N, k_1) & \cdots & V(k_N, k_N) \end{vmatrix} \tag{2.9}$$

is the Fredholm determinant of a linear integral operator $\hat{V}$ with the kernel

$$V(k, p) = \frac{e_+(k)e_-(p) - e_+(p)e_-(k)}{k - p} \tag{2.10}$$

defined on $[-1, 1] \times [-1, 1]$. The functions entering this kernel are defined as follows:

$$\begin{aligned} e_+(k) &= \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-\tau(k)/2}[(1 - \cos\eta)\,\mathrm{e}^{\tau(k)} E_0(k) + \sin\eta] \\ e_-(k) &= \frac{1}{\sqrt{\pi}} \mathrm{e}^{\tau(k)/2} \\ \tau(k) &= \mathrm{i}k^2 t - \mathrm{i}kx \\ E_0(k) &= \text{p.v.} \int_{-\infty}^{\infty} \mathrm{d}p \, \frac{\mathrm{e}^{-\tau(p)}}{\pi(p - k)}. \end{aligned} \tag{2.11}$$

Note that $V(k, p)$ is symmetric, $V(k, p) = V(p, k)$, and nonsingular at $k = p$.

To define B_{--} and b_{++} introduce the resolvent operator $\hat{R}$:

$$\hat{I} - \hat{R} = (\hat{I} + \hat{V})^{-1}. \tag{2.12}$$

Due to the specific form (2.10) of the kernel $V(k, p)$ the resolvent kernel $R(k, p)$ may be written as

$$R(k, p) = \frac{f_+(k) f_-(p) - f_+(p) f_-(k)}{k - p} \tag{2.13}$$

where functions $f_\pm$ are the solutions to the integral equations

$$f_\pm(k) + \int_{-1}^{1} \mathrm{d}p\, V(k, p) f_\pm(p) = e_\pm(k). \tag{2.14}$$

For the pairs $e_\pm(k)$ and $f_\pm(k)$ we will often use the vector notation:

$$\vec{e}(k) = \begin{pmatrix} e_+(k) \\ e_-(k) \end{pmatrix} \qquad \vec{f}(k) = \begin{pmatrix} f_+(k) \\ f_-(k) \end{pmatrix}. \tag{2.15}$$

Now define the functions B_{ab} and C_{ab}

$$B_{ab} = \int_{-1}^{1} \mathrm{d}k\, e_a(k) f_b(k) \qquad C_{ab} = \int_{-1}^{1} \mathrm{d}k\, k e_a(k) f_b(k) \tag{2.16}$$

where a and b run through two values: $a, b = \pm$. In particular,

$$B_{--} = \int_{-1}^{1} \mathrm{d}k\, e_-(k) f_-(k). \tag{2.17}$$

The function b_{++} in (2.7) is

$$b_{++} = B_{++} - (1 - \cos\eta) G_0 \tag{2.18}$$

where $G_0(x, t)$ is the vacuum Green function

$$G_0(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-\tau(k)} = \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}\pi/4}}{2\sqrt{\pi t}}\, \mathrm{e}^{\mathrm{i}x^2/4t} & t \neq 0 \\ \delta(x) & t = 0. \end{cases} \tag{2.19}$$

2.3. *Deformation of the integration contour in equations (2.6) and (2.7)*

In this subsection we examine the analytic properties of the functions entering the representations (2.6) and (2.7). Using these properties we transform the integrals in equations (2.6) and (2.7) to a form convenient for the asymptotical analysis.

Let us change the integration variable η in (2.6) and (2.7) to

$$z = \mathrm{e}^{\mathrm{i}\eta}. \tag{2.20}$$

Then

$$G_h(x, t) = \frac{\mathrm{e}^{-\mathrm{i}t}}{8\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{z} F(z) B_{--}(z) \det(\hat{I} + \hat{V})(z) \tag{2.21}$$

$$G_e(x, t) = \frac{\mathrm{e}^{\mathrm{i}t}}{\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{(z-1)^2} F(z) b_{++}(z) \det(\hat{I} + \hat{V})(z) \tag{2.22}$$

where

$$F(z) = 1 + \frac{z}{2 - z} + \frac{1}{2z - 1}. \tag{2.23}$$

The integration contour, $|z| = 1$, is oriented counterclockwise.

Introduce the function

$$E(k) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{e}^{-\tau(p)}}{p - k} \qquad \mathrm{Im}\, k \neq 0 \tag{2.24}$$

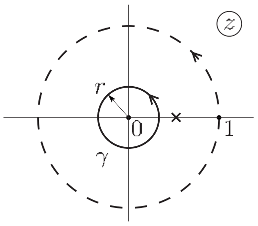

Figure 1. Deformation of the integration contour. The dashed circle shows the integration contour in equations (2.21) and (2.22); the solid circle of radius $r < 1/2$ shows the integration contour γ in equations (2.32) and (2.33). Both contours are oriented counterclockwise. The cross indicates the pole of the function $F(z)$ at $z = 1/2$.

and denote by $E_+(k)$ and $E_-(k)$ its analytic branches in the upper and lower half-planes, respectively. The analytic continuations of $E_+(k)$ and $E_-(k)$ to the real axis satisfy

$$E_+(k) + E_-(k) = E_0(k) \qquad E_+(k) - E_-(k) = \mathrm{i}\,\mathrm{e}^{-\tau(k)} \qquad k \in \mathbb{R}. \tag{2.25}$$

Using (2.25) the kernel $V(k, p) = V(k, p; z)$ can be rewritten as

$$V(k, p; z) = (z-1)[W_1(k, p) + z^{-1} W_2(k, p)] \tag{2.26}$$

where

$$W_1(k, p) = -\frac{1}{2\pi} \exp\left[\frac{\tau(k) + \tau(p)}{2}\right] \frac{E_+(k) - E_+(p)}{k - p} \tag{2.27}$$

$$W_2(k, p) = \frac{1}{2\pi} \exp\left[\frac{\tau(k) + \tau(p)}{2}\right] \frac{E_-(k) - E_-(p)}{k - p}. \tag{2.28}$$

By using the Fredholm theory of linear integral equations (see, e.g., [6], chapter 3, section 7) the following two facts may be shown: (a) the Fredholm determinant $\det(\hat{I} + \hat{V})(z)$ with the kernel $V(k, p; z)$ defined by (2.26)–(2.28) is analytic in the complex z-plane except at most at $z = 0$ and $z = \infty$. (b) The corresponding resolvent kernel $R(k, p; z)$ may be represented as

$$R(k, p; z) = (z-1) \frac{D(k, p; z)}{\det(\hat{I} + \hat{V})(z)} \tag{2.29}$$

where the function $D(k, p; z)$ is analytic in the complex z-plane except at most at $z = 0$ and $z = \infty$. The explicit form of $D(k, p; z)$ is given in the reference cited above. We will not use it in our calculations.

Consider equation (2.21). The function B_{--}, entering this equation, is defined by equation (2.17). Inverting (2.14)

$$e_-(k) - \int_{-1}^{1} \mathrm{d}p\, R(p, k; z) e_-(p) = f_-(k) \tag{2.30}$$

and substituting the resulting expression in (2.17), one gets

$$B_{--}(z) = \int_{-1}^{1} \mathrm{d}k\, e_-^2(k) - \int_{-1}^{1} \mathrm{d}k \int_{-1}^{1} \mathrm{d}p\, e_-(k) R(k, p; z) e_-(p). \tag{2.31}$$

Combining equations (2.29) and (2.31) one sees that $B_{--}(z)\det(\hat{I} + \hat{V})(z)$ is analytic in the complex z-plane except at most at $z = 0$ and $z = \infty$. Keeping this in mind, deform the integration contour in (2.21) to a contour γ shown in figure 1. Since the function $F(z)$ has a simple pole at the point $z = 1/2$, one gets

$$G_h(x, t) = \frac{\mathrm{e}^{-\mathrm{i}t}}{4} B_{--}(1/2) \det(\hat{I} + \hat{V})(1/2) + \frac{\mathrm{e}^{-\mathrm{i}t}}{8\pi \mathrm{i}} \oint_\gamma \frac{\mathrm{d}z}{z} F(z) B_{--}(z) \det(\hat{I} + \hat{V})(z). \tag{2.32}$$

Next consider equation (2.22). Like the function $B_{--}(z)\det(\hat{I}+\hat{V})(z)$, the function $b_{++}(z)\det(\hat{I}+\hat{V})(z)$ is analytic in the complex z-plane except at most at $z=0$ and $z=\infty$. Note that despite the singular factor $(z-1)^{-2}$ the integrand does not have a singularity at $z=1$ due to the second-order zero of $b_{++}(z)$ at this point. Indeed, one can see from (2.11) that $e_+\sim\eta$ for $\eta\sim 0$. Therefore, $f_+\sim\eta$ for $\eta\sim 0$, which is seen from equation (2.14). Using the definitions (2.16), (2.18) and (2.20) one can see that $b_{++}\sim(1-z)^2$ for $1-z\sim 0$. Deforming the integration contour in (2.22) to the contour γ shown in figure 1, one gets

$$G_e(x,t)=4\,\mathrm{e}^{\mathrm{i}t}b_{++}(1/2)\det(\hat{I}+\hat{V})(1/2)+\frac{\mathrm{e}^{\mathrm{i}t}}{\pi\mathrm{i}}\oint_\gamma\frac{\mathrm{d}z}{(z-1)^2}F(z)b_{++}(z)\det(\hat{I}+\hat{V})(z). \quad (2.33)$$

Consider a special case of $t=0$. The kernel (2.28) becomes equal to zero since $E_-(k)=0$. Hence, the kernel (2.26) becomes a linear function of z. This implies that (see, e.g., [6], chapter 3, section 7) (a) the determinant $\det(\hat{I}+\hat{V})(z)$ becomes an analytic function of z for all $z\in\mathbb{C}$ and (b) the function $D(k,p;z)$ in equation (2.12) also becomes an analytic function of z for all $z\in\mathbb{C}$. Thus, by virtue of (2.31) the function $B_{--}(z)\det(\hat{I}+\hat{V})(z)$ is analytic in the complex z-plane and the second term on the right-hand side of (2.32) vanishes:

$$G_h(x,0)=\tfrac{1}{4}B_{--}(1/2)\det(\hat{I}+\hat{V})(1/2). \quad (2.34)$$

The function $G_e(x,0)$ can be obtained from equation (2.34) using equation (2.4).

2.4. Differential equations

In this subsection we write differential equations for the Fredholm determinant $\det(\hat{I}+\hat{V})(x,t,\eta)$ entering the representations (2.6) and (2.7).

First, we obtain the differential equation with respect to η. As follows from (2.20) and (2.26)–(2.28) the η derivative of the kernel (2.10) is given by

$$\partial_\eta V(k,p)=\frac{\mathrm{i}}{1-\mathrm{e}^{-\mathrm{i}\eta}}V(k,p)+\frac{1-\mathrm{e}^{-\mathrm{i}\eta}}{\mathrm{i}}W_2(k,p). \quad (2.35)$$

Differentiating the identity

$$\ln\det(\hat{I}+\hat{V})=\operatorname{Tr}\ln(\hat{I}+\hat{V}) \quad (2.36)$$

we immediately get

$$\partial_\eta\ln\det(\hat{I}+\hat{V})(x,t,\eta)=\frac{\mathrm{i}}{1-\mathrm{e}^{-\mathrm{i}\eta}}\operatorname{Tr}[(\hat{I}+\hat{V})^{-1}\hat{V}]+\frac{1-\mathrm{e}^{-\mathrm{i}\eta}}{\mathrm{i}}\operatorname{Tr}[(\hat{I}+\hat{V})^{-1}\hat{W}_2]. \quad (2.37)$$

Transform the right-hand side of (2.37) employing (2.12), (2.13) and (2.15). The result for the first term is

$$\operatorname{Tr}[(\hat{I}+\hat{V})^{-1}\hat{V}]=\operatorname{Tr}\hat{R}=-\mathrm{i}\int_{-1}^{1}\mathrm{d}k\,[\vec{f}(k)]^T\sigma_2\partial_k\vec{f}(k) \quad (2.38)$$

where σ_2 is the second Pauli matrix

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad\sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\qquad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad (2.39)$$

The second term on the right-hand side of (2.37) is more complicated:

$$\begin{aligned}\operatorname{Tr}[(\hat{I}+\hat{V})^{-1}\hat{W}_2]&=\operatorname{Tr}[(\hat{I}-\hat{R})\hat{W}_2]\\&=\int_{-1}^{1}\mathrm{d}k\,W_2(k,k)-\int_{-1}^{1}\mathrm{d}k\int_{-1}^{1}\mathrm{d}p\,R(k,p)W_2(p,k).\end{aligned} \quad (2.40)$$

The differential equations for $\det(\hat{I}+\hat{V})(x,t,\eta)$ with respect to x and t can be obtained in a similar way. They read

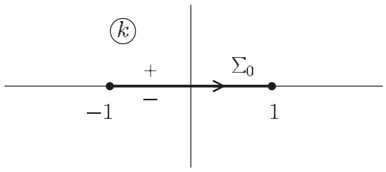

Figure 2. The conjugation contour Σ_0. The thick line connecting the points $k = -1$ and $k = 1$ shows the conjugation contour Σ_0 for the RH problem (2.52). The arrow shows the direction of the orientation of this contour.

$$\partial_x \ln\det(\hat{I} + \hat{V})(x, t, \eta) = \mathrm{i}B_{+-} = \mathrm{i}B_{-+} \tag{2.41}$$

$$\partial_t \ln\det(\hat{I} + \hat{V})(x, t, \eta) = -\mathrm{i}\,[C_{+-} + C_{-+} + (1 - \cos\eta)G_0 B_{--}]\,. \tag{2.42}$$

Finally, the following initial conditions for the differential equations follow from (2.10) and (2.11):

$$\det(\hat{I} + \hat{V})(x, t, 0) = 1 \tag{2.43}$$

and

$$\det(\hat{I} + \hat{V})(0, 0, \eta) = 1. \tag{2.44}$$

The differential equations (2.37), (2.41) and (2.42) will be used in the subsequent asymptotical calculation of $\det(\hat{I} + \hat{V})$.

2.5. Riemann–Hilbert problem

In this subsection we briefly recall the notion of the RH problem and its relation to the objects entering the representations (2.6) and (2.7).

The standard formulation of the Riemann–Hilbert problem is as follows [4, 5, 7]: for a given oriented contour Σ in the complex k-plane and for a given matrix $\chi(k)$ defined on this contour, seek a matrix $Y(k)$ such that

$$\begin{aligned} &\text{(a) } Y(k) \text{ is analytic in } \mathbb{C}\backslash\Sigma \\ &\text{(b) } Y_+(k) = Y_-(k)\chi(k) \qquad k \in \Sigma \\ &\text{(c) } Y(k) \to I \quad \text{as} \quad k \to \infty. \end{aligned} \tag{2.45}$$

The contour Σ is called the 'conjugation contour' or 'jump contour'; the matrix $\chi(k)$ is called the 'conjugation matrix' or 'jump matrix'. The matrices $Y_\pm(k)$ are the boundary values of $Y(k)$ as k approaches Σ:

$$Y_\pm(k) = \lim_{k'\to k} Y(k') \qquad \text{where} \quad k' \in \pm \text{ side of } \Sigma. \tag{2.46}$$

By convention, the + side (respectively, − side) of Σ lies to the left (respectively, right) as one traverses the contour in the direction of the orientation. An example of such an oriented contour is shown in figure 2.

By using the Cauchy transform one can see that the RH problem (2.45) is equivalent to the matrix-valued integral equation ([7], chapter XV)

$$Y(k) = I + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\mathrm{d}s}{s-k} Y_-(s)[\chi(s) - I]. \tag{2.47}$$

(The functions $Y_\pm(s)$, $\chi(s)$ and $Y_\pm(s)\chi(s)$ are supposed to be integrable along the contour Σ. We will impose this condition for all RH problems appearing below.) Suppose the jump matrix $\chi(k)$ depends on the external parameter, x, and

$$\chi(k; x) = I + \mathcal{O}(x^{-\alpha}) \qquad \alpha \geqslant 0 \quad \text{as} \quad x \to \infty \tag{2.48}$$

uniformly for $k \in \Sigma$. Then, it can be shown solving (2.47) perturbatively that

$$Y = I + \mathcal{O}(x^{-\alpha}) \qquad \text{as} \quad x \to \infty \tag{2.49}$$

uniformly outside an arbitrarily small vicinity of the contour Σ. We will use the estimate (2.49) in the subsequent asymptotical analysis.

Next we relate the RH problem (2.45) to the objects entering the determinant representations (2.6) and (2.7). Consider the following conjugation matrix

$$\nu_0(k; x, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\pi \mathrm{i} \begin{pmatrix} -e_-(k)e_+(k) & e_+^2(k) \\ -e_-^2(k) & e_+(k)e_-(k) \end{pmatrix} \tag{2.50}$$

or in the vector notation (2.15)

$$\nu_0(k; x, t) = I + 2\pi \vec{e}(k)[\vec{e}(k)]^T \sigma_2. \tag{2.51}$$

Choose the conjugation contour to be the interval $[-1, 1]$ oriented as shown in figure 2 and denote this contour as Σ_0. Let Y be a solution to the RH problem

$$\begin{aligned} &\text{(a) } Y(k) \text{ is analytic in } \mathbb{C}\backslash\Sigma_0 \\ &\text{(b) } Y_+(k) = Y_-(k)\nu_0(k; x, t) \qquad k \in \Sigma_0 \\ &\text{(c) } Y(k) \to I \quad \text{as} \quad k \to \infty. \end{aligned} \tag{2.52}$$

Then the components f_+ and f_- of the vector

$$\vec{f}(k) = Y_-(k)\vec{e}(k) = Y_+(k)\vec{e}(k) \qquad k \in \Sigma_0 \tag{2.53}$$

satisfy equations (2.14). Indeed, consider equation (2.47) where the conjugation matrix χ is chosen to be the matrix ν_0 given in equation (2.51). Multiply equation (2.47) by the vector $\vec{e}(k)$ from the right. Equations (2.14) follow immediately.

Combining equations (2.53) and (2.47) where the matrix χ is chosen to be ν_0 write

$$Y(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_{\Sigma_0} \frac{\mathrm{d}p}{p - k} \begin{pmatrix} e_-(p)f_+(p) & -e_+(p)f_+(p) \\ e_-(p)f_-(p) & -e_+(p)f_-(p) \end{pmatrix}. \tag{2.54}$$

Comparing equations (2.16) and (2.54) one sees that the functions B_{ab} and C_{ab} can be found from the large k expansion of $Y(k)$

$$Y(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} -B_{-+} & B_{++} \\ -B_{--} & B_{+-} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -C_{-+} & C_{++} \\ -C_{--} & C_{+-} \end{pmatrix} + \mathcal{O}\left(\frac{1}{k^3}\right). \tag{2.55}$$

Consider an arbitrary compact domain $\mathcal{D}$ in the complex plane of the parameter z (2.20), not containing the point $z = 0$. Since the Fredholm determinant in equations (2.21) and (2.22) is analytic in $\mathbb{C}\backslash\{0\}$, it only has a finite set of zeros $\{z_n\} = \{z_1, \ldots, z_n\}$ in $\mathcal{D}$. The integral equations (2.14) are solved uniquely for any $z \notin \{z_n\}$ in $\mathcal{D}$. The solutions $f_\pm$ of the integral equations (2.14) can be found from a solution $Y(k)$ of the RH problem (2.52) by using equation (2.53). The functions B_{ab} and C_{ab} can be found from the expansion (2.55). The Fredholm determinant $\det(\hat{I} + \hat{V})(z)$ is completely defined by the differential equations (2.41) and (2.42) with the initial condition (2.44) or, alternatively, by the differential equation (2.37) with the initial condition (2.43). Since the functions $B_{--}(z)\det(\hat{I} + \hat{V})(z)$ in (2.21) and $b_{++}(z)\det(\hat{I} + \hat{V})(z)$ in (2.22) are analytic in $\mathbb{C}\backslash\{0\}$, their values at the points $z_1, \ldots, z_n$ can be found by analytic continuation.

3. Riemann–Hilbert problem at $t = 0$

In this section we solve the RH problem (2.52) asymptotically for $x \to \infty$ and $t = 0$. As is obvious from (2.34), to calculate $G_h(x, 0)$ it is sufficient to solve this RH problem at a specific value $z = 1/2$ of the parameter z. Nevertheless, we will construct the solution for all complex z outside an arbitrarily small vicinity of the point $z = 0$. This general result will prove useful in the subsequent analysis of the $t \neq 0$ case.

In section 3.1 we formulate explicitly the RH problem (2.52) for $t = 0$, see equation (3.3). In sections 3.2 and 3.3, following along the lines of [4, 5], we transform the RH problem (3.3) so as to make it convenient for the large x analysis. In section 3.4 we give an explicit solution to the jump relation (3.3b) in the vicinity of the end points $k = \pm 1$ of the contour Σ_0. In section 3.5 we construct the solution to the RH problem (3.3) in the large x limit by matching the solution given in section 3.4 with the condition (3.3c). In section 3.6 we summarize the results.

3.1. Riemann–Hilbert problem (2.52) at $t = 0$

In this subsection we give an explicit formulation of the RH problem (2.52) at $t = 0$.

By taking the limit $t \to 0$ and assuming that $x > 0$ in (2.11) we get for $e_\pm(k)$

$$e_+(k) = \frac{\mathrm{i}}{2}\frac{1}{\sqrt{\pi}}\,\mathrm{e}^{\mathrm{i}kx/2}(1 - \mathrm{e}^{\mathrm{i}\eta}) \qquad e_-(k) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-\mathrm{i}kx/2}. \tag{3.1}$$

The conjugation matrix (2.50) becomes

$$\nu_0(k; x, 0) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\eta} & -2\mathrm{i}\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{e}^{\mathrm{i}\eta}\sin^2\frac{\eta}{2} \\ 2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}kx} & 2 - \mathrm{e}^{\mathrm{i}\eta} \end{pmatrix}. \tag{3.2}$$

The oriented contour Σ_0 shown in figure 2 remains unchanged. Thus, we have defined the RH problem for $t = 0$

$$\begin{aligned} &\text{(a) } Y(k) \text{ is analytic in } \mathbb{C}\backslash\Sigma_0 \\ &\text{(b) } Y_+(k) = Y_-(k)\nu_0(k; x, 0) \qquad k \in \Sigma_0 \\ &\text{(c) } Y(k) \to I \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.3}$$

Introduce a matrix

$$P = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\eta/2}\sin\frac{\eta}{2} \\ \mathrm{e}^{-\mathrm{i}\eta/2}\operatorname{cosec}\frac{\eta}{2} & 0 \end{pmatrix}. \tag{3.4}$$

Note that $P^2 = I$. The jump matrix (3.2) and the functions (3.1) satisfy the involutions

$$P\nu_0(k)P^{-1} = [\nu_0(-k)]^{-1} \tag{3.5}$$

and (note (2.15))

$$P\vec{e}(k) = \vec{e}(-k) \tag{3.6}$$

respectively. As a consequence, we have the involution

$$PY(-k)P^{-1} = Y(k) \tag{3.7}$$

for the solution $Y(k)$ of the RH problem (3.3) and, by virtue of (2.53), the involution

$$P\vec{f}(k) = \vec{f}(-k) \tag{3.8}$$

for $\vec{f}(k)$.

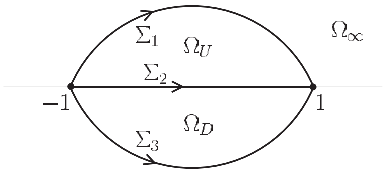

Figure 3. The lens-shaped contour Σ_{lens}. The thick lines show the lens-shaped contour $\Sigma_{\text{lens}} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. It is the conjugation contour for the RH problem (3.16). This contour divides the complex k-plane into three regions: Ω_U, Ω_D and Ω_∞.

3.2. Riemann–Hilbert problem on a lens-shaped contour

In this subsection an RH problem equivalent to the RH problem (3.3) is formulated on the lens-shaped conjugation contour shown in figure 3.

Represent $Y(k)$ as a product of two matrices

$$Y(k) = U(k)L(k) \tag{3.9}$$

and define $L(k)$ by the formula

$$L(k) = \begin{cases} I & k \in \Omega_\infty \\ \nu_1(k) & k \in \Omega_U \\ [\nu_3(k)]^{-1} & k \in \Omega_D. \end{cases} \tag{3.10}$$

A lens-shaped contour $\Sigma_{\text{lens}} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ dividing the complex k-plane into the regions Ω_U, Ω_D and Ω_∞ is displayed in figure 3. The matrices ν_1 and ν_3 are defined as

$$\nu_1(k;x) = \begin{pmatrix} 1 & -2\mathrm{i}\,\mathrm{e}^{\mathrm{i}kx}\sin^2\frac{\eta}{2} \\ 0 & 1 \end{pmatrix} \tag{3.11}$$

$$\nu_3(k;x) = \begin{pmatrix} 1 & 0 \\ 2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{e}^{-\mathrm{i}\eta} & 1 \end{pmatrix}. \tag{3.12}$$

Introducing a matrix ν_2 by the formula

$$\nu_2 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\eta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\eta} \end{pmatrix} \tag{3.13}$$

one can see that the following representation for the conjugation matrix (3.2) can be written:

$$\nu_0 = \nu_3\nu_2\nu_1. \tag{3.14}$$

Note that

$$P\nu_1(k)P^{-1} = [\nu_3(-k)]^{-1} \qquad P\nu_2P^{-1} = \nu_2^{-1} \tag{3.15}$$

where the matrix P is defined by (3.4).

It follows from equations (3.3), (3.9), (3.10) and (3.14) that $U(k)$ solves the following RH problem

$$\begin{aligned} &\text{(a) } U(k) \text{ is analytic in } \mathbb{C}\backslash\Sigma_{\text{lens}} \\ &\text{(b) } U_+(k) = U_-(k)\nu_i(k) \qquad k \in \Sigma_i \quad i = 1,2,3 \\ &\text{(c) } U(k) \to I \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.16}$$

We will calculate the large x asymptotics of $U(k)$; the asymptotics of $Y(k)$ will then follow from (3.9). The advantage of the RH problem (3.16) from the point of view of the large x analysis will be clear from the next subsection.

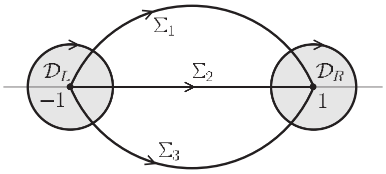

Figure 4. The contour used in the factorization of the RH problem (3.16). The discs $\mathcal{D}_R$ and $\mathcal{D}_L$ are shaded in grey; their boundaries $\partial\mathcal{D}_R$ and $\partial\mathcal{D}_L$ are oriented clockwise.

3.3. Factorization of the Riemann–Hilbert problem (3.16)

In this subsection we present the scheme of the analysis of the RH problem (3.16) in the large x limit.

The x-dependent entry of the jump matrix ν_1 (respectively, ν_3) entering (3.16) decays exponentially as $x \to \infty$ everywhere on the jump contour Σ_1 (respectively, Σ_3) except at the points $k = \pm 1$. The jump matrix ν_2 is a constant matrix. For the RH problems with such a behaviour of the jump matrices, the following scheme of the large x analysis can be employed [4, 5]. Denote by $\mathcal{V}$ a vicinity of the points $k = \pm 1$. Split $U(k)$ into a product of three matrices

$$U = KSQ. \tag{3.17}$$

The matrix $Q(k)$ satisfies the jump relation $Q_+(k) = Q_-(k)\nu_2$ everywhere on the contour Σ_2 and the jump relation (3.16b) on the part of contour Σ_{lens} lying in $\mathcal{V}$. The matrix $S(k)$ represents the corrections to $Q(k)$ due to the mismatch between the analytic branches of $Q(k)$ in and outside $\mathcal{V}$. The matrix $K(k)$ represents the corrections to $S(k)Q(k)$ due to the discontinuities, exponentially small as $x \to \infty$, across the parts of the contours Σ_1 and Σ_3 lying outside $\mathcal{V}$. Below we give the precise sense of Q, S and K.

Let $\mathcal{V}$ consist of two discs $\mathcal{D}_R$ and $\mathcal{D}_L$ shown in figure 4. Let the radii of these discs be equal and less than 1. Denote the domain $\mathbb{C}\backslash(\mathcal{D}_R \cup \mathcal{D}_L)$ as $\mathcal{D}_\infty$. Introduce a matrix $Q(k)$ by

$$Q(k) = \begin{cases} Q_R(k) & k \in \mathcal{D}_R \\ Q_L(k) & k \in \mathcal{D}_L \\ Q_\infty(k) & k \in \mathcal{D}_\infty \end{cases} \tag{3.18}$$

where the matrices Q_∞, Q_R and Q_L are defined as follows. The matrix $Q_\infty(k)$ is analytic everywhere in $\mathcal{D}_\infty$ except the contour $\Sigma_2 \cap \mathcal{D}_\infty$. On this contour it satisfies the jump relation with the jump matrix (3.13)

$$[Q_\infty(k)]_+ = [Q_\infty(k)]_-\nu_2 \qquad k \in \Sigma_2 \cap \mathcal{D}_\infty \tag{3.19}$$

and is normalized at infinity:

$$Q_\infty(k) \to I \qquad \text{as} \quad k \to \infty. \tag{3.20}$$

We choose the following (obviously, not unique) solution to (3.19) and (3.20):

$$Q_\infty(k) = \begin{pmatrix} \left(\frac{k-1}{k+1}\right)^{\frac{\eta}{2\pi}} & 0 \\ 0 & \left(\frac{k-1}{k+1}\right)^{-\frac{\eta}{2\pi}} \end{pmatrix}. \tag{3.21}$$

Note that it satisfies the involution

$$Q_\infty(k) = PQ_\infty(-k)P^{-1}. \tag{3.22}$$

The matrix $Q_R(k)$ is analytic everywhere in $\mathcal{D}_R$ except the contour $\Sigma_{\text{lens}} \cap \mathcal{D}_R$. On this contour it satisfies the jump relation with the jump matrices (3.11)–(3.13)

$$[Q_R(k)]_+ = [Q_R(k)]_-\nu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_R \quad i = 1, 2, 3. \tag{3.23}$$

A solution to the jump relation (3.23) is given in section 3.4. The matrix $Q_L(k)$ is analytic everywhere in $\mathcal{D}_L$, except the contour $\Sigma_{\rm lens} \cap \mathcal{D}_L$. On this contour it satisfies a jump relation similar to (3.23)

$$[Q_L(k)]_+ = [Q_L(k)]_- \nu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_L \quad i = 1, 2, 3. \tag{3.24}$$

A solution to the jump relation (3.24) is given in section 3.4.

It should be stressed that any solution to the jump relation (3.23) (respectively, (3.24)) multiplied from the left by an arbitrary matrix with the entries being entire functions of k in $\mathcal{D}_R$ (respectively, $\mathcal{D}_L$), still remains a solution to (3.23) (respectively, (3.24)). In subsequent calculations we will employ this property to simplify the form of the conjugation matrices θ_R and θ_L defined by equation (3.27).

Next define the matrix S entering the decomposition (3.17) as a solution to the following RH problem:

$$\begin{aligned} &\text{(a) } S(k) \text{ is analytic in } \mathbb{C}\backslash(\partial\mathcal{D}_R \cup \partial\mathcal{D}_L) \\ &\text{(b) } S_+(k) = S_-(k)\theta_{R,L}(k) \qquad k \in \partial\mathcal{D}_{R,L} \\ &\text{(c) } S(k) \to I \quad \text{as} \quad k \to \infty \end{aligned} \tag{3.25}$$

with the clockwise oriented conjugation contours $\partial\mathcal{D}_{R,L}$ shown in figure 4. The jump matrices $\theta_{R,L}(k)$ are chosen to ensure the continuity of SQ across $\partial\mathcal{D}_{R,L}$:

$$S_+(k)Q_+(k) = S_-(k)Q_-(k) \qquad k \in \partial\mathcal{D}_{R,L}. \tag{3.26}$$

This implies

$$\theta_{R,L}(k) = Q_{R,L}(k)[Q_\infty(k)]^{-1} \qquad k \in \partial\mathcal{D}_{R,L}. \tag{3.27}$$

We solve the RH problem (3.25) in the large x limit in section 3.5.

With the matrices Q and S defined above one can easily see that the matrix K in the decomposition (3.17) solves the following RH problem:

$$\begin{aligned} &\text{(a) } K(k) \text{ is analytic in } \mathbb{C}\backslash(\Sigma_{1,3} \cap \mathcal{D}_\infty) \\ &\text{(b) } K_+(k) = K_-(k)[S(k)Q(k)]\nu_i(k)[S(k)Q(k)]^{-1} \qquad k \in \Sigma_i \cap \mathcal{D}_\infty \quad i = 1, 3 \\ &\text{(c) } K(k) \to I \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.28}$$

We solve the RH problem (3.28) in the large x limit in section 3.6.

3.4. Exact solutions to the jump relations (3.23) and (3.24)

In this subsection we explicitly construct matrices Q_R and Q_L satisfying the jump relations (3.23) and (3.24), respectively.

Since the conjugation matrix (2.50) is 2π-periodic in $\mathrm{Re}\,\eta$ we will restrict our considerations to the interval

$$-\pi \leqslant \mathrm{Re}\,\eta \leqslant \pi. \tag{3.29}$$

To shorten notation, introduce the parameter

$$\epsilon = 1 - \frac{|\mathrm{Re}\,\eta|}{\pi}. \tag{3.30}$$

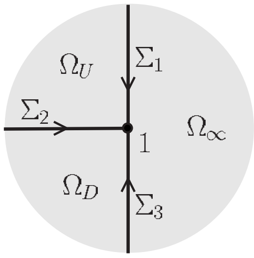

Figure 5. The jump contour for the jump relation (3.23). The disc $\mathcal{D}_R$ is shaded in grey. The segments of the contours Σ_1 and Σ_3 lying in $\mathcal{D}_R$ are straight and perpendicular to Σ_2.

Consider the part of the contour Σ_{lens} lying in the disc $\mathcal{D}_R$. For convenience of presentation we choose Σ_1 and Σ_3 to be perpendicular to Σ_2, as shown in figure 5. Consider a matrix $\Phi_R(k;x)$ with the entries

$$\begin{aligned}
\Phi^R_{11}(k;x) &= x^{-\frac{\eta}{2\pi}}\varsigma_1(k-1)\,\mathrm{e}^{\frac{\mathrm{i}\eta}{4}}\,\Psi\left[-\frac{\eta}{2\pi},1;-\mathrm{i}x(k-1)\right]\\
\Phi^R_{21}(k;x) &= -x^{\frac{\eta}{2\pi}}\frac{\pi\,\varsigma_1(k-1)\,\mathrm{e}^{-\frac{\mathrm{i}\eta}{4}}\,\mathrm{e}^{-\mathrm{i}x}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\sin^2\frac{\eta}{2}}\Psi\left[1-\frac{\eta}{2\pi},1;-\mathrm{i}x(k-1)\right]\\
\Phi^R_{12}(k;x) &= x^{-\frac{\eta}{2\pi}}\frac{\pi\,\varsigma_2(k-1)\,\mathrm{e}^{\frac{3\mathrm{i}\eta}{4}}\,\mathrm{e}^{\mathrm{i}x}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)}\Psi\left[1+\frac{\eta}{2\pi},1;\mathrm{i}x(k-1)\right]\\
\Phi^R_{22}(k;x) &= x^{\frac{\eta}{2\pi}}\varsigma_2(k-1)\,\mathrm{e}^{\frac{\mathrm{i}\eta}{4}}\,\Psi\left[\frac{\eta}{2\pi},1;\mathrm{i}x(k-1)\right].
\end{aligned}\tag{3.31}$$

Here $\Psi(a,b;w)$ is the Tricomi Ψ function, which has a branch cut discontinuity across the negative real axis [8]

$$\begin{aligned}
&\Psi(a,1,w+\mathrm{i}0)-[1-2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}a\pi}\sin(\pi a)]\Psi(a,1,w-\mathrm{i}0)\\
&\qquad=\frac{2\pi\mathrm{i}\,\mathrm{e}^{-\mathrm{i}a\pi}}{\Gamma^2(a)}e^{w}\Psi(1-a,1,-w)\qquad w\in\mathbb{R}_-
\end{aligned}\tag{3.32}$$

and $\varsigma_{1,2}(w)$ are piecewise-constant functions defined by

$$\varsigma_1(w)=\begin{cases}\mathrm{e}^{-\mathrm{i}\eta} & -\pi<\arg w<-\dfrac{\pi}{2}\\ 1 & -\dfrac{\pi}{2}<\arg w<\pi\end{cases}\tag{3.33}$$

and

$$\varsigma_2(w)=\begin{cases}1 & -\pi<\arg w<\dfrac{\pi}{2}\\ \mathrm{e}^{-\mathrm{i}\eta} & \dfrac{\pi}{2}<\arg w<\pi.\end{cases}\tag{3.34}$$

Using (3.32) it is easy to check that $\Phi_R(k;x)$ satisfies the jump relation (3.23) on the contour $\Sigma_{\text{lens}}\cap\mathcal{D}_R$ shown in figure 5

$$[\Phi_R(k;x)]_+=[\Phi_R(k;x)]_-\nu_i(k;x)\qquad k\in\Sigma_i\cup\mathcal{D}_R\quad i=1,2,3.\tag{3.35}$$

It follows from the involution (3.15) that a solution $\Phi_L(k)$ to the jump relation (3.24) may be obtained from $\Phi_R(k)$ via the involution

$$\Phi_L(k;x)=P\Phi_R(-k;x)P^{-1}\qquad k\in\mathcal{D}_L.\tag{3.36}$$

As was mentioned in the paragraph following equation (3.24), any matrix Q_R of the form

$$Q_R(k)=E(k)\Phi_R(k)\tag{3.37}$$

with $E(k)$ being an arbitrary analytic in $\mathcal{D}_R$ matrix, satisfies (3.23). Choosing

$$E(k) = \begin{pmatrix} (k+1)^{-\frac{\eta}{2\pi}} & 0 \\ 0 & (k+1)^{\frac{\eta}{2\pi}} \end{pmatrix} \tag{3.38}$$

one ensures that $Q_R(k)$ goes to $Q_\infty(k)$, as $x \to \infty$, for any $\epsilon \neq 0$, see (3.41). This will facilitate the asymptotic solution of the RH problem (3.25). Taking into account (3.36) and (3.37) define $Q_L(k)$ in $\mathcal{D}_L$ by the formula

$$Q_L(k) = P Q_R(-k) P^{-1} \qquad k \in \mathcal{D}_L. \tag{3.39}$$

To calculate the large x asymptotics of $Q_R(k)$ we use the asymptotic formula for the Tricomi Ψ function ([8], section 6.13.1)

$$\Psi(a, 1, w) = w^{-a}\left[1 - \frac{a^2}{w} + \mathcal{O}\left(\frac{1}{w^2}\right)\right] \qquad \text{as} \quad w \to \infty. \tag{3.40}$$

Applying (3.40) to (3.31) one gets from (3.37):

$$Q_R(k) = Q_\infty(k) + \frac{1}{x(k-1)} \begin{pmatrix} -\mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \left(\frac{k-1}{k+1}\right)^{\frac{\eta}{2\pi}} & a_R x^{-\frac{\eta}{\pi}} (k^2-1)^{-\frac{\eta}{2\pi}} \\ b_R x^{\frac{\eta}{\pi}} (k^2-1)^{\frac{\eta}{2\pi}} & \mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \left(\frac{k+1}{k-1}\right)^{\frac{\eta}{2\pi}} \end{pmatrix} + \mathcal{O}(x^{-1-\epsilon}) \tag{3.41}$$

where

$$a_R = -\frac{\mathrm{i}\pi\, \mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\, \mathrm{e}^{\mathrm{i}x}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)} \qquad b_R = -\frac{\mathrm{i}\pi\, \mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\, \mathrm{e}^{-\mathrm{i}x}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\sin^2\frac{\eta}{2}}. \tag{3.42}$$

The large x asymptotics of $Q_L(k)$ follows from (3.39) and (3.41). Note that the terms in asymptotic expansions of Q_R explicitly written down in (3.41) do not have jumps across the contours Σ_1 and Σ_3. This reflects the fact that the x-dependent entries in the jump matrices ν_1 and ν_3 are exponentially small on Σ_1 and Σ_3, respectively.

3.5. Solution to the Riemann–Hilbert problem (3.25) in the large x limit

In this subsection we solve the RH problem (3.25) in the large x limit.

Consider the conjugation matrices θ_R and θ_L of the RH problem (3.25). They are defined by formula (3.27) and satisfy the involution

$$\theta_L(k) = P\theta_R(-k)P^{-1} \qquad k \in \partial\mathcal{D}_L \tag{3.43}$$

as can be shown using the involutions (3.22) and (3.39). Substituting the asymptotic expression (3.41) into (3.27) one gets the approximation of the matrix θ_R by a matrix $\tilde{\theta}_R$

$$\theta_R = \tilde{\theta}_R[I + \mathcal{O}(x^{-1-\epsilon})] \tag{3.44}$$

uniform for $k \in \partial\mathcal{D}_R$. Here

$$\tilde{\theta}_R(k) = I + \frac{\upsilon_R(k)}{x(k-1)} \qquad k \in \partial\mathcal{D}_R \tag{3.45}$$

and

$$\upsilon_R(k) = \begin{pmatrix} -\mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 & a_R[x(k+1)]^{-\frac{\eta}{\pi}} \\ b_R[x(k+1)]^{\frac{\eta}{\pi}} & \mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \end{pmatrix}. \tag{3.46}$$

It follows from the involution (3.43) that the matrix θ_L is approximated by a matrix $\tilde{\theta}_L$

$$\theta_L = \tilde{\theta}_L[I + \mathcal{O}(x^{-1-\epsilon})] \tag{3.47}$$

uniformly for $k \in \partial\mathcal{D}_L$, where

$$\tilde{\theta}_L(k) = P\tilde{\theta}_R(-k)P^{-1} \qquad k \in \partial\mathcal{D}_L. \tag{3.48}$$

Consider the RH problem

$$\begin{aligned} &\text{(a) } \tilde{S}(k) \text{ is analytic in } \mathbb{C}\backslash(\partial\mathcal{D}_R \cup \partial\mathcal{D}_L) \\ &\text{(b) } \tilde{S}_+(k) = \tilde{S}_-(k)\tilde{\theta}_{R,L}(k) \qquad k \in \partial\mathcal{D}_{R,L} \\ &\text{(c) } \tilde{S}(k) \to I \qquad \text{as} \quad k \to \infty \end{aligned} \tag{3.49}$$

with the matrices $\tilde{\theta}_R$ and $\tilde{\theta}_L$ defined by equations (3.45) and (3.48), respectively. Comparing (3.44) and (3.47) with (2.48) and (2.49) we conclude that the solution $\tilde{S}(k)$ of the RH problem (3.49) approximates the solution $S(k)$ of the RH problem (3.25)

$$S = \tilde{S}[I + \mathcal{O}(x^{-1-\epsilon})] \tag{3.50}$$

uniformly outside a vicinity of the conjugation contours $\partial\mathcal{D}_R$ and $\partial\mathcal{D}_L$.

Due to the simple analytical structure of the conjugation matrices (3.45) and (3.48) the RH problem (3.49) can be solved exactly. To begin with, introduce the following notation for the analytic branches of $\tilde{S}(k)$

$$\tilde{S}(k) = \begin{cases} \tilde{S}_R(k) & k \in \mathcal{D}_R \\ \tilde{S}_L(k) & k \in \mathcal{D}_L \\ \tilde{S}_\infty(k) & k \in \mathcal{D}_\infty. \end{cases} \tag{3.51}$$

The matrix $\tilde{\theta}_R(k)$ analytically continued from $\partial\mathcal{D}_R$ into $\mathcal{D}_R$ has a simple pole at $k = 1$, as follows from the representation (3.45). The matrix $\tilde{S}_R(k)$ is analytic in $\mathcal{D}_R$. Therefore, the jump relation (3.49b)

$$\tilde{S}_\infty(k) = \tilde{S}_R(k)\tilde{\theta}_R(k) \qquad k \in \partial\mathcal{D}_R \tag{3.52}$$

implies that the analytic continuation of $\tilde{S}_\infty(k)$ into $\mathcal{D}_R$ has a simple pole at $k = 1$. Similarly, the analytic continuation of $\tilde{S}_\infty(k)$ into $\mathcal{D}_L$ has a simple pole at $k = -1$. Finally, taking into account the condition given in equation (3.49c), $\tilde{S}_\infty(\infty) = I$, we see that the most general form of $\tilde{S}_\infty(k)$ is

$$\tilde{S}_\infty(k) = I + \frac{A_R}{k-1} + \frac{A_L}{k+1} \qquad k \in \mathbb{C} \tag{3.53}$$

with the matrices A_R and A_L being independent of k.

The involution

$$P\tilde{S}_\infty(k)P^{-1} = \tilde{S}_\infty(-k) \tag{3.54}$$

which follows from the involution (3.48), provides a simple relation between A_R and A_L

$$A_R = -PA_LP^{-1} \tag{3.55}$$

so we only need to calculate one of the matrices, say, A_R. For this, invert (3.52):

$$\tilde{S}_R(k) = \tilde{S}_\infty(k)[\tilde{\theta}_R(k)]^{-1} \qquad k \in \partial\mathcal{D}_R. \tag{3.56}$$

Using (3.45) and (3.53) the right-hand side of equation (3.56) can be analytically continued into $\mathcal{D}_R$. In the following, equation (3.56) will be understood to be such a continuation. The matrix $[\tilde{\theta}_R(k)]^{-1}$ has the form

$$[\tilde{\theta}_R(k)]^{-1} = [\det\tilde{\theta}_R(k)]^{-1}\left[I - \frac{\upsilon_R(k)}{x(k-1)}\right] \tag{3.57}$$

where

$$\det \tilde{\theta}_R(k) = 1 + \frac{\det \upsilon_R}{x^2(k-1)^2} \tag{3.58}$$

and $\det \upsilon_R$ is a k-independent number

$$\det \upsilon_R = \left(\frac{\eta}{2\pi}\right)^4 + \left(\frac{\eta}{2\pi}\right)^2. \tag{3.59}$$

The right-hand side of (3.56) must be analytic in $\mathcal{D}_R$, since the left-hand side is. The matrix $[\tilde{\theta}_R(k)]^{-1}$ goes to zero as $k \to 1$, cancelling the pole of $\tilde{S}_\infty(k)$ at this point. However, $[\tilde{\theta}_R(k)]^{-1}$ is singular at the points where

$$\det \tilde{\theta}_R(k) = 0. \tag{3.60}$$

Equation (3.60) is quadratic in k. Its solutions, $k^R_{1,2}$, lie inside the disc $\mathcal{D}_R$ for sufficiently large x. Therefore, to cancel the singularities in (3.56) the matrix A_R has to satisfy the pair of matrix equations

$$\left(I + \frac{A_R}{k^R_{1,2} - 1} - \frac{P^{-1} A_R P}{k^R_{1,2} + 1}\right)\left[I - \frac{\upsilon_R\left(k^R_{1,2}\right)}{x\left(k^R_{1,2} - 1\right)}\right] = 0. \tag{3.61}$$

The determinant of the matrix in square brackets in (3.61) is equal to $\det \tilde{\theta}_R\left(k^R_{1,2}\right)$ and, therefore, is equal to zero. Employing this fact one gets (after some algebra) the matrix A_R. The large x asymptotics of A_R is

$$A_R = \frac{2}{\kappa}\begin{pmatrix} \varkappa_L^2 - \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2 & -\mathrm{i}\varkappa_R\, \mathrm{e}^{\mathrm{i}\frac{\eta}{2}} \sin\frac{\eta}{2} \\ -\mathrm{i}\varkappa_L\, \mathrm{e}^{-\mathrm{i}\frac{\eta}{2}} \operatorname{cosec}\frac{\eta}{2} & \varkappa_R^2 + \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2 \end{pmatrix} + \mathcal{O}(x^{-1-\epsilon}) \tag{3.62}$$

where

$$\varkappa_R = \frac{\pi\, \mathrm{e}^{\mathrm{i}x}(2x)^{-1-\frac{\eta}{\pi}}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)\sin\frac{\eta}{2}} \qquad \varkappa_L = \frac{\pi\, \mathrm{e}^{-\mathrm{i}x}(2x)^{-1+\frac{\eta}{\pi}}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\sin\frac{\eta}{2}} \tag{3.63}$$

and

$$\kappa = 1 + \varkappa_R^2 + \varkappa_L^2. \tag{3.64}$$

Applying the involution (3.55) to (3.62) one gets the large x asymptotics of A_L:

$$A_L = \frac{2}{\kappa}\begin{pmatrix} -\varkappa_R^2 - \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2 & \mathrm{i}\varkappa_L\, \mathrm{e}^{\mathrm{i}\frac{\eta}{2}} \sin\frac{\eta}{2} \\ \mathrm{i}\varkappa_R\, \mathrm{e}^{-\mathrm{i}\frac{\eta}{2}} \operatorname{cosec}\frac{\eta}{2} & -\varkappa_L^2 + \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2 \end{pmatrix} + \mathcal{O}(x^{-1-\epsilon}). \tag{3.65}$$

3.6. Riemann–Hilbert problem (3.3): results in the large x limit

In this subsection we show that the matrix K in the decomposition (3.17) does not contribute to the asymptotic solution of the RH problem (3.16) to the order we are interested in. This will conclude the asymptotic solution of the RH problem (3.3).

Consider the decomposition (3.17). In section 3.5 we constructed explicitly the approximation $\tilde{S}$ to S up to the order of $x^{-1-\epsilon}$. This approximation is uniform outside an arbitrarily small vicinity of the conjugation contours $\partial\mathcal{D}_R$ and $\partial\mathcal{D}_L$. The matrix Q was constructed explicitly in sections 3.3 and 3.4. Using these results, solve asymptotically the RH problem (3.28) for the matrix K. The matrices ν_1 and ν_3, entering (3.28b), converge as $x \to \infty$ to the identity matrix uniformly and exponentially fast on the contours $\Sigma_1 \cap \mathcal{D}_\infty$ and $\Sigma_3 \cap \mathcal{D}_\infty$, respectively. One can prove that

$$K = I + o(x^{-\infty}) \tag{3.66}$$

uniformly outside a vicinity of the contours $\Sigma_{1,3} \cap \mathcal{D}_\infty$. The basic idea in obtaining (3.66) is the same as in obtaining (3.50): to use (2.47). One should take into account, though, that we have now, in contrast to (3.50), no uniform estimate for the jump matrix in (3.28b) in a vicinity of the contours $\partial\mathcal{D}_{R,L}$ since we have no uniform estimate for $S(k)$ entering this matrix, in this vicinity. For further details we refer the reader to [4].

Substituting (3.50) and (3.66) into the decomposition (3.17) one gets the estimate

$$U = \tilde{S}Q[I + \mathcal{O}(x^{-1-\epsilon})] \tag{3.67}$$

uniform outside an arbitrarily small vicinity of the contours $\partial\mathcal{D}_{R,L}$ and $\Sigma_{1,3} \cap \mathcal{D}_\infty$. Using (3.9) one gets the approximation to $Y(k)$ up to the order of $x^{-1-\epsilon}$, uniform outside this vicinity. This completes the solution to the RH problem (3.3) in the large x limit.

4. Equal-time correlation functions

In this section we calculate the equal-time correlation functions $G_h(x,0)$ and $G_e(x,0)$ in the large x limit. The asymptotic expressions for these functions were obtained in [9, 10]. The main purpose of this section is to give a detailed illustration of our method on this known example. The procedures of this section will be extensively used in the subsequent analysis of the time-dependent correlation functions.

In section 4.1 we calculate the functions B_{ab} (2.16) from the large k expansion (2.55) of $Y(k)$. From B_{+-} thus obtained we calculate the Fredholm determinant $\det(\hat{I}+\hat{V})$ in the large x limit, see equation (4.6). The constant $C(\eta)$ entering (4.6) is calculated in section 4.2. We give the answer for $G_h(x,0)$ and $G_e(x,0)$ in section 4.3.

4.1. Large x asymptotics of B_{ab} and of the Fredholm determinant $\det(\hat{I}+\hat{V})$

In this subsection we use the results of section 3 on the asymptotic solution of the RH problem (3.3) to calculate the functions B_{ab} and the Fredholm determinant $\det(\hat{I}+\hat{V})$ in the large x limit.

It follows from (3.67), (3.9), (3.18) and (3.51) that in the vicinity of $k=\infty$ the following uniform approximation is valid:

$$Y = \tilde{S}_\infty Q_\infty + \mathcal{O}(x^{-1-\epsilon}). \tag{4.1}$$

Recall that the matrix $Q_\infty(k)$ is given by formula (3.21), the matrix $\tilde{S}_\infty(k)$ by formulae (3.53), (3.62) and (3.65). The approximation (4.1) being uniform, the functions B_{ab} can be calculated from the large k expansion (2.55) of equation (4.1). Expanding (3.21) and (3.53) in inverse powers of k and comparing with (2.55) one gets

$$\begin{pmatrix} -B_{-+} & B_{++} \\ -B_{--} & B_{+-} \end{pmatrix} = A_R + A_L - \frac{\eta}{\pi}\sigma_3 + \mathcal{O}(x^{-1-\epsilon}) \tag{4.2}$$

where σ_3 is the third Pauli matrix (2.39). Substituting equations (3.62) and (3.65) into (4.2) one gets for B_{+-} and B_{--}

$$B_{+-} = \frac{\eta}{\pi} + \frac{2}{\kappa}\left[\varkappa_R^2 - \varkappa_L^2 + \frac{\mathrm{i}}{x}\left(\frac{\eta}{2\pi}\right)^2\right] + \mathcal{O}(x^{-1-\epsilon}) \tag{4.3}$$

$$B_{--} = 2\mathrm{i}\,\mathrm{e}^{-\frac{\mathrm{i}\eta}{2}}\operatorname{cosec}\frac{\eta}{2}(\varkappa_L - \varkappa_R)[1 + \mathcal{O}(x^{-\epsilon})] \tag{4.4}$$

where $\varkappa_R$ and $\varkappa_L$ are defined by (3.63) and κ is defined by (3.64).

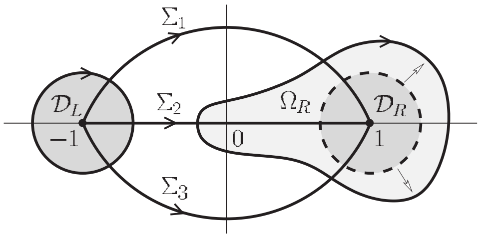

Figure 6. Analytic continuation. Equation (3.67) is analytically continued from the disc $\mathcal{D}_R$ into the region Ω_R containing the interval [0, 1].

To calculate the Fredholm determinant $\det(\hat{I} + \hat{V})$ we integrate (2.41) taking into account the boundary condition (2.44):

$$\ln\det(\hat{I} + \hat{V}) = \mathrm{i}\int_0^x \mathrm{d}y\, B_{+-}(y). \tag{4.5}$$

Substituting equation (4.3) into (4.5) and calculating the integral in the large x limit, one obtains

$$\ln\det(\hat{I} + \hat{V}) = \frac{\mathrm{i}\eta}{\pi}x - 2\left(\frac{\eta}{2\pi}\right)^2 \ln x + C(\eta) + \mathcal{O}(x^{-\epsilon}). \tag{4.6}$$

We calculate the integration constant $C(\eta)$ in section 4.2.

4.2. *Calculation of $C(\eta)$*

In this subsection we calculate the constant $C(\eta)$ in (4.6) using the differential equation (2.37).

Consider equation (2.37). For $t = 0$ the kernel $W_2(k, p)$ vanishes and, therefore, the second term on the right-hand side of (2.37) vanishes. The remaining term is given by equation (2.38). With the help of the involution (3.8) and the identity

$$P^T\sigma_2 P = -\sigma_2 \tag{4.7}$$

the integral in (2.38) can be reduced to the integral over positive k, yielding

$$\partial_\eta \ln\det(\hat{I} + \hat{V})(x, 0, \eta) = \frac{2}{1 - \mathrm{e}^{-\mathrm{i}\eta}}\int_0^1 \mathrm{d}k[\vec{f}(k)]^T\sigma_2\partial_k\vec{f}(k). \tag{4.8}$$

The large x asymptotics of the function $\vec{f}(k)$ in equation (4.8) is calculated as follows. Use equation (2.53)

$$\vec{f}(k) = Y_+(k)\vec{e}(k) \qquad k \in \Sigma_2 \tag{4.9}$$

and replace $Y(k)$ by its large x asymptotics obtained in section 3. It follows from (3.9) that

$$Y_+(k) = U_+(k)\nu_1(k) \qquad k \in \Sigma_2 \tag{4.10}$$

where $U(k)$ solves the RH problem (3.16). Formula (3.67) gives the approximation to $U(k)$ uniform outside an arbitrarily small vicinity of the contours $\partial\mathcal{D}_{R,L}$ and $\Sigma_{1,3} \cap \mathcal{D}_\infty$. Since the radius of $\mathcal{D}_R$ was chosen initially to be less than 1, the contour $\partial\mathcal{D}_R$ crosses the interval [0, 1]. The integration variable k in (4.8) lies in this interval and, therefore, the estimate (3.67) cannot be applied everywhere on the integration contour. This problem is circumvented by the analytic continuation of equation (3.67) from the disc $\mathcal{D}_R$ into a bigger region Ω_R containing the interval [0, 1], as shown in figure 6. The matrices $Q_R(k)$ and $\tilde{S}_R(k)$ are given in Ω_R by the same formulae as in $\mathcal{D}_R$: the matrix Q_R by (3.37), (3.38) and (3.31); the matrix $\tilde{S}_R$ by (3.56). Using (4.9) and (4.10) one gets the approximation to $\vec{f}(k)$

$$\vec{f} = \tilde{S}_R \vec{g}[I + \mathcal{O}(x^{-1-\epsilon})] \tag{4.11}$$

uniform for $k \in [0, 1]$. The function $\vec{g}$ is defined by the equation

$$\vec{g}(k) = \begin{pmatrix} g_+(k) \\ g_-(k) \end{pmatrix} = [Q_R(k)]_+ \nu_1(k)\vec{e}(k) \qquad k \in [0, 1]. \tag{4.12}$$

The matrix ν_1 entering (4.12) is given by equation (3.11), the function $\vec{e}$ by (2.15) and (3.1), the matrix $Q_R(k)$ by (3.31), (3.37) and (3.38). With the help of the identity [8]

$${}_1F_1(a, b; w) = \frac{\Gamma(b)}{\Gamma(b-a)} \mathrm{e}^{\mathrm{i}\pi a\lambda} \Psi(a, b; w) + \frac{\Gamma(b)}{\Gamma(a)} \mathrm{e}^{\mathrm{i}\pi(a-b)\lambda} \mathrm{e}^{w} \Psi(b-a, b; -w) \tag{4.13}$$

where $\lambda = \operatorname{sgn} \operatorname{Im} w$, one gets for (4.12)

$$\begin{aligned} g_+(k) &= -\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{\mathrm{i}\eta}{4}}}{\Gamma\left(-\frac{\eta}{2\pi}\right)} \mathrm{e}^{-\frac{\mathrm{i}kx}{2}+\mathrm{i}x} x^{-\frac{\eta}{2\pi}} (k+1)^{-\frac{\eta}{2\pi}} {}_1F_1\left[\frac{\eta}{2\pi}+1, 1; \mathrm{i}x(k-1)\right] \\ g_-(k) &= \frac{\sqrt{\pi}\,\mathrm{e}^{-\frac{\mathrm{i}\eta}{4}}}{\Gamma\left(\frac{\eta}{2\pi}\right)\sin\frac{\eta}{2}} \mathrm{e}^{-\frac{\mathrm{i}kx}{2}} x^{\frac{\eta}{2\pi}} (k+1)^{\frac{\eta}{2\pi}} {}_1F_1\left[\frac{\eta}{2\pi}, 1; \mathrm{i}x(k-1)\right]. \end{aligned} \tag{4.14}$$

Using the uniform estimate (4.11) and formulae (4.14) one can derive the following estimate for (4.8)

$$\partial_\eta \ln\det(\hat{I} + \hat{V}) = \frac{2}{1 - \mathrm{e}^{-\mathrm{i}\eta}} \int_0^1 \mathrm{d}k [\vec{g}(k)]^T \sigma_2 \partial_k \vec{g}(k)[1 + \mathcal{O}(x^{-1-\epsilon})]. \tag{4.15}$$

Performing the integral in (4.15) asymptotically one gets

$$\partial_\eta \ln\det(\hat{I} + \hat{V}) = \frac{\mathrm{i}x}{\pi} - \frac{\eta}{\pi^2}\ln x + C_1(\eta) + \mathcal{O}(x^{-\epsilon}) \tag{4.16}$$

where

$$C_1(\eta) = \frac{\eta}{2\pi^2}\left[\psi\left(\frac{\eta}{2\pi}+1\right) + \psi\left(-\frac{\eta}{2\pi}+1\right) - 2(\ln 2 + 1)\right] \tag{4.17}$$

and $\psi(w)$ is the digamma function,

$$\psi(w) = \frac{\mathrm{d}}{\mathrm{d}w}\ln\Gamma(w). \tag{4.18}$$

Integrating equation (4.16) over η, taking into account the initial condition (2.43) and comparing the resulting expression with (4.6) one gets

$$C(\eta) = \frac{\eta}{\pi}\ln\left[\frac{\Gamma\left(1+\frac{\eta}{2\pi}\right)}{\Gamma\left(1-\frac{\eta}{2\pi}\right)}\right] - \frac{\eta^2}{2\pi^2}(\ln 2 + 1) - \frac{1}{\pi}\int_0^\eta \mathrm{d}\lambda \ln\left[\frac{\Gamma\left(1+\frac{\lambda}{2\pi}\right)}{\Gamma\left(1-\frac{\lambda}{2\pi}\right)}\right]. \tag{4.19}$$

4.3. Equal-time correlation functions: the results

In this subsection we give the answer for the correlation functions $G_h(x, 0)$ and $G_e(x, 0)$ in the large x limit.

The correlation function $G_h(x, 0)$ is given by formula (2.34). The asymptotics of B_{--} is given by (4.4); the asymptotics of $\det(\hat{I} + \hat{V})$ by (4.6) with the constant $C(\eta)$ given by (4.19). Recall that $z = \exp(\mathrm{i}\eta)$, therefore the point $z = 1/2$ corresponds to $\eta = \mathrm{i}\ln 2$. The final answer for $G_h(x, 0)$ is

$$G_h(x, 0) = \Xi x^{-\alpha} \sin\left(x - x_0 - \frac{\ln 2}{\pi}\ln x\right) \exp\left(-\frac{\ln 2}{\pi}x\right) \tag{4.20}$$

where the anomalous exponent α is given by

$$\alpha = 1 - \frac{1}{2}\left(\frac{\ln 2}{\pi}\right)^2 \tag{4.21}$$

the phase shift x_0 by

$$x_0 = \frac{(\ln 2)^2}{\pi} - 2\,\mathrm{Im}\left[\ln\Gamma\left(\frac{\mathrm{i}\ln 2}{2\pi}\right)\right] \tag{4.22}$$

and the constant Ξ by

$$\Xi = -4\pi\sqrt{2}\exp\left\{C(\mathrm{i}\ln 2) - 2\,\mathrm{Re}\left[\ln\Gamma\left(\frac{\mathrm{i}\ln 2}{2\pi}\right)\right]\right\} \tag{4.23}$$

where $C(\eta)$ is defined by equation (4.19). The relative correction to (4.20) is of the order of x^{-1}. Note that equations (4.20) and (4.21) were first derived in [9]. The phase shift (4.22) and the constant (4.23) were calculated in [10].

The asymptotics of the function $G_e(x, 0)$ is obtained from equation (4.20) by using equation (2.4). Recall that in the derivation of equation (4.20) we assumed that x is positive. The result for negative x follows from equation (2.1).

5. Riemann–Hilbert problem at $t \neq 0$. Space-like region

In this section we find the asymptotic solution of the Riemann–Hilbert problem (2.52) for $|x| > 2|t|$. We assume that $x > 0$ and $t > 0$. We will consider the large x asymptotics for a fixed value of

$$k_s = \frac{x}{2t}. \tag{5.1}$$

In section 5.1 we reformulate the RH problem (2.52) so as to make it similar to the RH problem (3.3). In sections 5.2 and 5.3 we repeat the construction made for the $t = 0$ case in sections 3.2 and 3.3, respectively. In section 5.4 we give an explicit solution to the jump relation (5.10b) in a vicinity of the end points $k = \pm 1$ of the contour Σ_0. In section 5.5 we perform a matching procedure analogous to that of section 3.5. In section 5.6 we summarize the results.

5.1. Reformulation of the Riemann–Hilbert problem (2.52)

In this subsection we reformulate the RH problem (2.52) so as to make it similar to the RH problem (3.3).

Represent $Y(k)$ as a product of two matrices

$$Y(k) = Z(k)V(k) \tag{5.2}$$

and define $V(k)$ by

$$V(k) = \begin{pmatrix} 1 & -2\sin^2\frac{\eta}{2}[E(k) - \mathrm{i}\,\mathrm{e}^{-\tau(k)}\varrho(k)] \\ 0 & 1 \end{pmatrix} \tag{5.3}$$

with $E(k)$ given by equation (2.24) and $\varrho(k)$ being a piecewise-constant function

$$\varrho(k) = \begin{cases} 1 & k \in \mathcal{S}_{\mathrm{I}} \\ -1 & k \in \mathcal{S}_{\mathrm{III}} \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

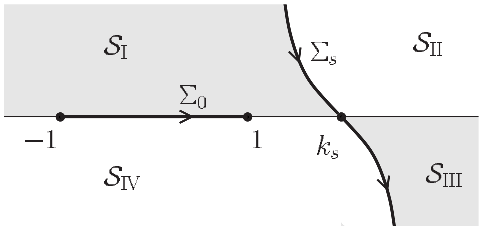

Figure 7. The conjugation contour $\Sigma_0 \cup \Sigma_s$ for the RH problem (5.10). The sectors $\mathcal{S}_{\mathrm{I}}$ and $\mathcal{S}_{\mathrm{III}}$ are shaded in grey.

The sectors $\mathcal{S}_{\mathrm{I}}$ and $\mathcal{S}_{\mathrm{III}}$ are shown in figure 7. The oriented contour Σ_s separating the sector $\mathcal{S}_{\mathrm{I}}$ from $\mathcal{S}_{\mathrm{II}}$ and the sector $\mathcal{S}_{\mathrm{III}}$ from $\mathcal{S}_{\mathrm{IV}}$ is chosen so that (a) it passes through the point k_s and (b) the real part of $\tau(k)$ is positive for all $k \in \Sigma_s$ except $k = k_s$

$$\operatorname{Re}\tau(k) > 0 \qquad k \in \Sigma_s \backslash \{k_s\} \tag{5.5}$$

and

$$\operatorname{Re}\tau(k) \to +\infty \qquad \text{as} \qquad k \to \infty \quad k \in \Sigma_s. \tag{5.6}$$

It is easy to see that $V(k)$ is analytic everywhere in the complex k-plane except the contour Σ_s, where it satisfies the jump relation

$$V_+(k) = [\mu_s(k)]^{-1} V_-(k) \qquad k \in \Sigma_s \tag{5.7}$$

with

$$\mu_s(k) = \begin{pmatrix} 1 & 2\mathrm{i}\,\mathrm{e}^{-\tau(k)}\sin^2\frac{\eta}{2} \\ 0 & 1 \end{pmatrix}. \tag{5.8}$$

Note that condition (5.6) implies

$$V(k) \to I \qquad \text{as} \quad k \to \infty. \tag{5.9}$$

With $V(k)$ and Σ_s defined above, one gets the following RH problem for $Z(k)$

$$\begin{aligned} &\text{(a) } Z(k) \text{ is analytic in } \mathbb{C}\backslash(\Sigma_0 \cup \Sigma_s) \\ &\text{(b) } Z_+(k) = Z_-(k)\mu_i(k) \qquad k \in \Sigma_i \quad i = 0, s \\ &\text{(c) } Z(k) \to I \quad \text{as} \quad k \to \infty \end{aligned} \tag{5.10}$$

where the contours Σ_0 and Σ_s are shown in figure 7. Since $Y(k)$ is analytic across Σ_s, the matrix μ_s should satisfy (5.7) and, therefore, is given by (5.8). The matrix μ_0 can be obtained comparing the jump relations (2.52b) and (5.10b)

$$\mu_0(k) = V(k)\nu_0(k; x, t)[V(k)]^{-1} \qquad k \in \Sigma_0. \tag{5.11}$$

Representing (2.50) in the form

$$\nu_0(k; x, t) = \begin{pmatrix} 1 & 2\sin^2\frac{\eta}{2}E_-(k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\eta} & 0 \\ 2\mathrm{i}\,\mathrm{e}^{\tau(k)} & \mathrm{e}^{-\mathrm{i}\eta} \end{pmatrix} \begin{pmatrix} 1 & -2\sin^2\frac{\eta}{2}E_+(k) \\ 0 & 1 \end{pmatrix} \tag{5.12}$$

with $E_\pm$ given by (2.24) and (2.25) one gets

$$\mu_0(k) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\eta} & -2\mathrm{i}\,\mathrm{e}^{-\tau(k)}\,\mathrm{e}^{\mathrm{i}\eta}\sin^2\frac{\eta}{2} \\ 2\mathrm{i}\,\mathrm{e}^{\tau(k)} & 2 - \mathrm{e}^{\mathrm{i}\eta} \end{pmatrix}. \tag{5.13}$$

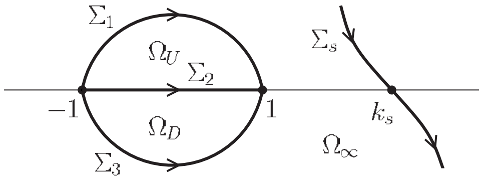

Figure 8. The conjugation contour for the RH problem (5.22). The contour consists of the lens-shaped contour $\Sigma_{\text{lens}} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and the contour Σ_s.

We see that the matrix μ_0 is different from the matrix ν_0, defined by (3.2), only in that $\mathrm{i}kx$ in the latter is replaced by $-\tau(k)$ in the former. The matrix μ_0 obeys the involution

$$P\mu_0(k; x, t)P^{-1} = [\mu_0(-k; x, -t)]^{-1} \tag{5.14}$$

with the matrix P defined by equation (3.4). This involution is a natural generalization of the involution (3.5).

5.2. *Riemann–Hilbert problem on the contour* $\Sigma_{\text{lens}} \cup \Sigma_s$

In this subsection an RH problem equivalent to the RH problem (5.10) is formulated on the conjugation contour $\Sigma_{\text{lens}} \cup \Sigma_s$. The procedure will resemble that of section 3.2.

Represent $Z(k)$ as a product of two matrices

$$Z(k) = U(k)L(k) \tag{5.15}$$

and define $L(k)$ by the formula

$$L(k) = \begin{cases} I & k \in \Omega_\infty \\ \mu_1(k) & k \in \Omega_U \\ [\mu_3(k)]^{-1} & k \in \Omega_D. \end{cases} \tag{5.16}$$

The lens-shaped contour $\Sigma_{\text{lens}} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is the same as in section 3.2. It is shown in figure 8. The matrices μ_1 and μ_3 are defined as

$$\mu_1(k) = \begin{pmatrix} 1 & -2\mathrm{i}\,\mathrm{e}^{-\tau(k)} \sin^2 \frac{\eta}{2} \\ 0 & 1 \end{pmatrix} \tag{5.17}$$

$$\mu_3(k) = \begin{pmatrix} 1 & 0 \\ 2\mathrm{i}\,\mathrm{e}^{\tau(k)}\,\mathrm{e}^{-\mathrm{i}\eta} & 1 \end{pmatrix}. \tag{5.18}$$

For the conjugation matrix (5.13) the following representation can be written:

$$\mu_0 = \mu_3\mu_2\mu_1 \tag{5.19}$$

where the matrix

$$\mu_2 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\eta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\eta} \end{pmatrix} \tag{5.20}$$

coincides with the matrix ν_2 defined by equation (3.13). Note that

$$P\mu_1(k; x, t)P^{-1} = [\mu_3(-k; x, -t)]^{-1} \qquad P\mu_2P^{-1} = \mu_2^{-1} \tag{5.21}$$

where the matrix P is defined by (3.4).

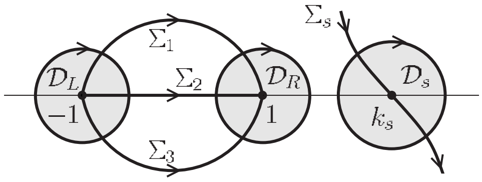

Figure 9. The contour used in the factorization of the RH problem (5.22). The discs $\mathcal{D}_R$, $\mathcal{D}_L$ and $\mathcal{D}_s$ are shaded in grey.

It follows from equations (5.10), (5.15), (5.16) and (5.19) that $U(k)$ solves the following RH problem:

$$\begin{aligned} &\text{(a) } U(k) \text{ is analytic in } \mathbb{C}\backslash(\Sigma_{\text{lens}} \cup \Sigma_s) \\ &\text{(b) } U_+(k) = U_-(k)\mu_i(k) \qquad k \in \Sigma_i \quad i = 1, 2, 3, s \\ &\text{(c) } U(k) \to I \qquad \text{as} \quad k \to \infty \end{aligned} \tag{5.22}$$

where the contours Σ_i are shown in figure 8. We will calculate the large x asymptotics of $U(k)$; the asymptotics of $Z(k)$ will then follow from (5.15).

5.3. Factorization of the Riemann–Hilbert problem (5.22)

In this subsection we present the scheme of the analysis of the RH problem (5.22) in the large x limit. The scheme is similar to that given in section 3.3.

Like the $t = 0$ case, the jump matrix μ_1 (respectively, μ_3) goes to the identity matrix, as $x \to \infty$, everywhere on the contour Σ_1 (respectively, Σ_3) except at the points $k = \pm 1$. The jump matrix μ_s goes to the identity matrix everywhere on Σ_s except at the point $k = k_s$. Therefore, to construct a uniform approximation to $U(k)$ we use the same scheme as in section 3.3. Introduce three discs $\mathcal{D}_R$, $\mathcal{D}_L$ and $\mathcal{D}_s$ as shown in figure 9. Denote the domain $\mathbb{C}\backslash(\mathcal{D}_R \cup \mathcal{D}_L \cup \mathcal{D}_s)$ as $\mathcal{D}_\infty$. Represent the solution $U(k)$ of the RH problem (5.22) as a product of three matrices

$$U = KSQ. \tag{5.23}$$

The matrix $Q(k)$ is given by the formula

$$Q(k) = \begin{cases} Q_{R,L}(k) & k \in \mathcal{D}_{R,L} \\ Q_s(k) & k \in \mathcal{D}_s \\ Q_\infty(k) & k \in \mathcal{D}_\infty. \end{cases} \tag{5.24}$$

The matrix Q_∞ is given by equation (3.21). The matrix Q_s is defined in $\mathcal{D}_s$ and satisfies

$$[Q_s(k)]_+ = [Q_s(k)]_- \mu_s(k) \qquad k \in \Sigma_s \cap \mathcal{D}_s. \tag{5.25}$$

We choose the following solution of the jump relation (5.25):

$$Q_s(k) = Q_\infty(k) \begin{pmatrix} 1 & 2\sin^2\frac{\eta}{2}\int_{\Sigma_s}\frac{\mathrm{d}p}{2\pi}\frac{\mathrm{e}^{-\tau(p)}}{p-k} \\ 0 & 1 \end{pmatrix} \tag{5.26}$$

where the integral runs along the contour Σ_s shown in figure 9. The matrices Q_R and Q_L are defined in the discs $\mathcal{D}_R$ and $\mathcal{D}_L$, respectively, and satisfy there the jump relations

$$[Q_R(k)]_+ = [Q_R(k)]_- \mu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_R \quad i = 1, 2, 3 \tag{5.27}$$

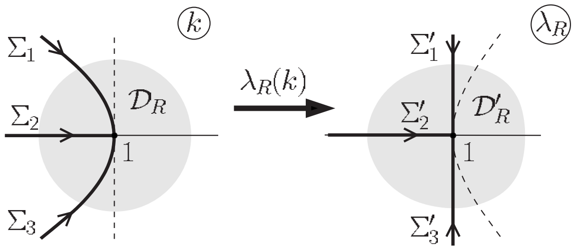

Figure 10. Conformal map λ_R. The disc $\mathcal{D}_R$ and its image $\mathcal{D}_R'$ are shaded in grey. The conjugation contours Σ_1 and Σ_3 are chosen so that their images Σ_1' and Σ_3' are parallel to the imaginary axis.

and

$$[Q_L(k)]_+ = [Q_L(k)]_- \mu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_L \quad i = 1, 2, 3. \tag{5.28}$$

We give solutions to the jump relations (5.27) and (5.28) in section 5.4.

The matrix S is defined as a solution of the RH problem

$$\begin{aligned} &\text{(a) } S(k) \text{ is analytic in } \mathbb{C}\backslash(\partial\mathcal{D}_R \cup \partial\mathcal{D}_L \cup \partial\mathcal{D}_s) \\ &\text{(b) } S_+(k) = S_-(k)\theta_\alpha(k) \qquad k \in \partial\mathcal{D}_\alpha \quad \alpha = R, L, s \\ &\text{(c) } S(k) \to I \quad \text{as} \quad k \to \infty \end{aligned} \tag{5.29}$$

with the clockwise oriented conjugation contours $\partial\mathcal{D}_{R,L,s}$ shown in figure 9 and the conjugation matrices $\theta_{R,L,s}$ defined by

$$\theta_\alpha(k) = Q_\alpha(k)[Q_\infty(k)]^{-1} \qquad k \in \partial\mathcal{D}_\alpha \quad \alpha = R, L, s. \tag{5.30}$$

We solve the RH problem (5.29) in the large x limit in section 5.5.

The matrix K in the decomposition (5.23) is constructed in complete analogy with the matrix K considered in section 3.3.

5.4. Exact solutions to the jump relations (5.27) and (5.28)

In this subsection we solve explicitly the jump relations (5.27) and (5.28) by mapping them on the jump relations (3.23) and (3.24), solved in section 3.4.

Introduce the 'light cone' variables x_R and x_L

$$x_R = x - 2t \qquad x_L = x + 2t. \tag{5.31}$$

Consider the jump relation (5.27) and make the following conformal map:

$$\lambda_R(k) = 1 - \frac{\tau(k) - \tau(1)}{\mathrm{i}x_R} \qquad k \in \mathcal{D}_R \tag{5.32}$$

of the disc $\mathcal{D}_R$ in the complex k-plane onto a vicinity $\mathcal{D}_R'$ of the point $\lambda_R = 1$ in the λ_R-plane, as shown in figure 10. Introducing a k-independent matrix T by the formula

$$T = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}t/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}t/2} \end{pmatrix} \tag{5.33}$$

one can easily check that

$$\mu_i(k) = T^{-1}\nu_i[\lambda_R(k); x_R]T \qquad k \in \mathcal{D}_R \quad i = 1, 2, 3. \tag{5.34}$$

Here the matrices μ_1, μ_3 and μ_2 are defined by formulae (5.17), (5.18) and (5.20), respectively; the matrices ν_1, ν_3 and ν_2 are defined by formulae (3.11), (3.12) and (3.13), respectively.

Choose the shape of the contours Σ_1 and Σ_3 in the k-plane so that their images Σ_1' and Σ_3' in the λ_R-plane are parallel to the imaginary axis, as shown in figure 10. Write equation (3.35) with k replaced by λ_R and x replaced by x_R:

$$[\Phi_R(\lambda_R; x_R)]_+ = [\Phi_R(\lambda_R; x_R)]_- \nu_i(\lambda_R; x_R) \qquad \lambda_R \in \Sigma_i' \quad i = 1, 2, 3 \tag{5.35}$$

where Φ_R is defined by (3.31). By virtue of (5.34) the jump relation (5.35) is equivalent to

$$\{\Phi_R[\lambda_R(k); x_R]T\}_+ = \{\Phi_R[\lambda_R(k); x_R]T\}_- \mu_i(k) \qquad k \in \Sigma_i \quad i = 1, 2, 3. \tag{5.36}$$

One sees that the matrix $\Phi_R[\lambda_R(k); x_R]T$ solves the jump relation (5.27) in the disc $\mathcal{D}_R$. Now we are in a position to write down an ansatz for $Q_R(k)$:

$$Q_R(k) = E_R(k)\Phi_R[\lambda_R(k); x_R]T \tag{5.37}$$

where

$$E_R(k) = Q_\infty(k) \begin{pmatrix} [\lambda_R(k) - 1]^{-\frac{\eta}{2\pi}} & 0 \\ 0 & [\lambda_R(k) - 1]^{\frac{\eta}{2\pi}} \end{pmatrix} T^{-1}. \tag{5.38}$$

With this choice of $E_R(k)$ the matrix $Q_R(k)$ goes to $Q_\infty(k)$, as $x \to \infty$, for any $\epsilon \neq 0$.

Taking into account the involution (5.21) choose the following solution to the jump relation (5.28)

$$Q_L(k; x, t) = P Q_R(-k; x, -t) P^{-1} \qquad k \in \mathcal{D}_L \tag{5.39}$$

where Q_R is given by formulae (5.37) and (5.38).

5.5. Solution to the Riemann–Hilbert problem (5.29) in the large x limit

In this subsection we solve the RH problem (5.29) in the large x limit. The procedure will resemble that of section 3.5.

Consider the conjugation matrices θ_R and θ_L defined by equation (5.30). They satisfy the involution

$$\theta_L(k; x, t) = P\theta_R(-k; x, -t)P^{-1} \qquad k \in \partial\mathcal{D}_L \tag{5.40}$$

as can be shown using the involutions (3.22) and (5.39). Substituting equations (5.37) and (5.38) into equation (5.30) and using the results of section 3.4 one gets the following approximation of the matrix θ_R by a matrix $\tilde{\theta}_R$:

$$\theta_R = \tilde{\theta}_R[I + \mathcal{O}(x^{-1-\epsilon})] \tag{5.41}$$

uniform for $k \in \partial\mathcal{D}_R$. Here

$$\tilde{\theta}_R(k) = I + \frac{\upsilon_R(k)}{x(k-1)} \qquad k \in \partial\mathcal{D}_R. \tag{5.42}$$

The matrix υ_R is given by

$$\upsilon_R(k) = \begin{pmatrix} -\mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \frac{2k_s}{2k_s-k-1} & a_R[x(k+1)]^{-\frac{\eta}{\pi}}\left(\frac{2k_s}{2k_s-k-1}\right)^{1+\frac{\eta}{\pi}} \\ b_R[x(k+1)]^{\frac{\eta}{\pi}}\left(\frac{2k_s}{2k_s-k-1}\right)^{1-\frac{\eta}{\pi}} & \mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \frac{2k_s}{2k_s-k-1} \end{pmatrix} \tag{5.43}$$

where

$$a_R = -\frac{\mathrm{i}\pi\, \mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\, \mathrm{e}^{\mathrm{i}(x-t)}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)} \qquad b_R = -\frac{\mathrm{i}\pi\, \mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\, \mathrm{e}^{-\mathrm{i}(x-t)}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\sin^2\frac{\eta}{2}}. \tag{5.44}$$

It follows from the involution (5.40) that the matrix θ_L is approximated by a matrix $\tilde{\theta}_L$

$$\theta_L = \tilde{\theta}_L[I + \mathcal{O}(x^{-1-\epsilon})] \tag{5.45}$$

uniformly for $k \in \partial\mathcal{D}_L$, where

$$\tilde{\theta}_L(k; x, t) = P\tilde{\theta}_R(-k; x, -t)P^{-1} = I + \frac{\upsilon_L(k)}{x(k+1)} \qquad k \in \partial\mathcal{D}_L \tag{5.46}$$

and

$$\upsilon_L(k; x, t) = -P\upsilon_R(-k; x, -t)P^{-1}. \tag{5.47}$$

The explicit expression for υ_L reads

$$\upsilon_L(k) = \begin{pmatrix} -\mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \frac{2k_s}{2k_s-k+1} & a_L[x(1-k)]^{\frac{\eta}{\pi}}\left(\frac{2k_s}{2k_s-k+1}\right)^{1-\frac{\eta}{\pi}} \\ b_L[x(1-k)]^{-\frac{\eta}{\pi}}\left(\frac{2k_s}{2k_s-k+1}\right)^{1+\frac{\eta}{\pi}} & \mathrm{i}\left(\frac{\eta}{2\pi}\right)^2 \frac{2k_s}{2k_s-k+1} \end{pmatrix} \tag{5.48}$$

where

$$a_L = \frac{\mathrm{i}\pi\,\mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\,\mathrm{e}^{-\mathrm{i}(x+t)}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)} \qquad b_L = \frac{\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\,\mathrm{e}^{\mathrm{i}(x+t)}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)\sin^2\frac{\eta}{2}}. \tag{5.49}$$

To calculate the conjugation matrix θ_s in the large x limit use the large x asymptotics of the integral in (5.26)

$$\int_{\Sigma_s} \frac{\mathrm{d}p}{2\pi}\frac{\mathrm{e}^{-\tau(p)}}{p-k} = -\frac{G_0}{k-k_s} + \mathcal{O}(x^{-3/2}) \tag{5.50}$$

where G_0 is defined by equation (2.19). Note that the leading term in (5.50) is of the order of $x^{-1/2}$. Substituting equation (5.50) into equation (5.26) and using equation (5.30) one gets

$$\theta_s = \tilde{\theta}_s[I + \mathcal{O}(x^{-3/2})] \tag{5.51}$$

uniformly for $k \in \partial\mathcal{D}_s$, where

$$\tilde{\theta}_s(k) = I + \frac{\upsilon_s(k)}{k-k_s} \qquad k \in \partial\mathcal{D}_s \tag{5.52}$$

and

$$\upsilon_s(k) = \begin{pmatrix} 0 & -2G_0\sin^2\frac{\eta}{2}\left(\frac{k-1}{k+1}\right)^{\frac{\eta}{\pi}} \\ 0 & 0 \end{pmatrix}. \tag{5.53}$$

Consider the RH problem

$$\begin{aligned} &\text{(a) } \tilde{S}(k) \text{ is analytic in } \mathbb{C}\backslash(\partial\mathcal{D}_R \cup \partial\mathcal{D}_L \cup \partial\mathcal{D}_s) \\ &\text{(b) } \tilde{S}_+(k) = \tilde{S}_-(k)\tilde{\theta}_\alpha(k) \qquad k \in \partial\mathcal{D}_\alpha \qquad \alpha = R, L, s \\ &\text{(c) } \tilde{S}(k) \to I \quad \text{as} \quad k \to \infty \end{aligned} \tag{5.54}$$

with the jump matrices $\tilde{\theta}_R, \tilde{\theta}_L$ and $\tilde{\theta}_s$ defined by (5.42), (5.46) and (5.52), respectively. In complete analogy with equation (3.50) the solution $\tilde{S}(k)$ of the RH problem (5.54) approximates the solution $S(k)$ of the RH problem (5.29)

$$S(k) = \tilde{S}(k)[I + \mathcal{O}(x^{-1-\epsilon})] \tag{5.55}$$

uniformly outside a vicinity of the conjugation contours $\partial\mathcal{D}_R$, $\partial\mathcal{D}_L$ and $\partial\mathcal{D}_s$.

To solve the RH problem (5.54) represent $\tilde{S}(k)$ as a product of two matrices

$$\tilde{S}(k) = M(k)N(k) \tag{5.56}$$

where the matrix $N(k)$ is the solution to the RH problem

$$\begin{aligned} &\text{(a) } N(k) \text{ is analytic in } \mathbb{C}\backslash(\partial\mathcal{D}_R \cup \partial\mathcal{D}_L) \\ &\text{(b) } N_+(k) = N_-(k)\tilde{\theta}_{R,L}(k) \qquad k \in \partial\mathcal{D}_{R,L} \\ &\text{(c) } N(k) \to I \quad \text{as} \quad k \to \infty \end{aligned} \tag{5.57}$$

with the jump matrices $\tilde{\theta}_R$ and $\tilde{\theta}_L$ given by (5.42) and (5.46), respectively. Comparing (5.54) and (5.57) one sees that the matrix $M(k)$ solves the following RH problem:

$$\begin{aligned}&\text{(a) } M(k) \text{ is analytic in } \mathbb{C}\backslash\partial\mathcal{D}_s\\ &\text{(b) } M_+(k) = M_-(k)N(k)\tilde{\theta}_s(k)N^{-1}(k) \qquad k\in\partial\mathcal{D}_s\\ &\text{(c) } M(k)\to I \quad \text{as} \quad k\to\infty\end{aligned} \tag{5.58}$$

where the jump matrix $\tilde{\theta}_s$ is defined by equation (5.52).

Let us solve the RH problem (5.57) first. Since the jump matrices (5.42) and (5.46) have the same analytic properties in the complex k-plane as the matrices (3.45) and (3.48), the RH problem (5.57) can be solved similarly as the RH problem (3.49). Introduce the following notation for the analytic branches of $N(k)$:

$$N(k) = \begin{cases} N_R(k) & k\in\mathcal{D}_R\\ N_L(k) & k\in\mathcal{D}_L\\ N_\infty(k) & k\in\mathbb{C}\backslash(\mathcal{D}_R\cup\mathcal{D}_L).\end{cases} \tag{5.59}$$

Using the same arguments as in deriving equation (3.53) from (3.52) one gets

$$N_\infty(k) = I + \frac{A_R}{k-1} + \frac{A_L}{k+1} \qquad k\in\mathbb{C} \tag{5.60}$$

where the matrices A_R and A_L do not depend on k. Dropping terms of order $x^{-1-\epsilon}$ and higher, we find

$$A_R = \frac{2}{\kappa}\begin{pmatrix} \varkappa_L^2 - \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2\frac{k_s}{k_s-1} & -\mathrm{i}\varkappa_R\,\mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\sin\frac{\eta}{2}\,\mathrm{e}^{-\mathrm{i}t}\left(\frac{k_s+1}{k_s-1}\right)^{\frac{\pi+\eta}{2\pi}}\\ -\mathrm{i}\varkappa_L\,\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\operatorname{cosec}\frac{\eta}{2}\,\mathrm{e}^{\mathrm{i}t}\left(\frac{k_s+1}{k_s-1}\right)^{\frac{\pi-\eta}{2\pi}} & \varkappa_R^2 + \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2\frac{k_s}{k_s-1}\end{pmatrix} \tag{5.61}$$

and

$$A_L = \frac{2}{\kappa}\begin{pmatrix} -\varkappa_R^2 - \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2\frac{k_s}{k_s+1} & \mathrm{i}\varkappa_L\,\mathrm{e}^{\mathrm{i}\frac{\eta}{2}}\sin\frac{\eta}{2}\,\mathrm{e}^{-\mathrm{i}t}\left(\frac{k_s+1}{k_s-1}\right)^{-\frac{\pi-\eta}{2\pi}}\\ \mathrm{i}\varkappa_R\,\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}\operatorname{cosec}\frac{\eta}{2}\,\mathrm{e}^{\mathrm{i}t}\left(\frac{k_s+1}{k_s-1}\right)^{-\frac{\pi+\eta}{2\pi}} & -\varkappa_L^2 + \frac{\mathrm{i}}{2x}\left(\frac{\eta}{2\pi}\right)^2\frac{k_s}{k_s+1}\end{pmatrix}. \tag{5.62}$$

Here

$$\varkappa_R = \frac{\pi\,\mathrm{e}^{\mathrm{i}x}(2x)^{-1-\frac{\eta}{\pi}}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)\sin\frac{\eta}{2}}\left(\frac{k_s^2}{k_s^2-1}\right)^{\frac{\pi+\eta}{2\pi}} \qquad \varkappa_L = \frac{\pi\,\mathrm{e}^{-\mathrm{i}x}(2x)^{-1+\frac{\eta}{\pi}}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\sin\frac{\eta}{2}}\left(\frac{k_s^2}{k_s^2-1}\right)^{\frac{\pi-\eta}{2\pi}} \tag{5.63}$$

and κ is defined by equation (3.64). The matrices $N_R(k)$ and $N_L(k)$ are found from (5.57b)

$$N_R(k) = N_\infty(k)[\tilde{\theta}_R(k)]^{-1} \qquad N_L(k) = N_\infty(k)[\tilde{\theta}_L(k)]^{-1}. \tag{5.64}$$

Next, solve the RH problem (5.58). Use the same method as was exploited in solving the RH problems (5.57) and (3.49). Denote the analytic branches of $M(k)$ by

$$M(k) = \begin{cases} M_s(k) & k\in\mathcal{D}_s\\ M_\infty(k) & k\in\mathbb{C}\backslash\mathcal{D}_s.\end{cases} \tag{5.65}$$

The jump matrix $N_\infty\tilde{\theta}_s N_\infty^{-1}$ in (5.58b) has a simple pole at $k = k_s$. Therefore

$$M_\infty(k) = I + \frac{A_s}{k-k_s} \qquad k\in\mathbb{C} \tag{5.66}$$

where the matrix A_s does not depend on k. Choosing

$$A_s = N_\infty(k_s)\upsilon_s(k_s)[N_\infty(k_s)]^{-1} \tag{5.67}$$

one solves (5.58). This can be checked directly.

5.6. Riemann–Hilbert problem (2.52): results in the large x limit

Like the $t = 0$ case, the matrix K in the decomposition (5.23) does not contribute to the asymptotic solution of the RH problem (5.22) to the order we are interested in. Substituting (5.55) into the decomposition (5.23) one gets the estimate

$$U = \tilde{S}Q[I + \mathcal{O}(x^{-1-\epsilon})] \tag{5.68}$$

uniform outside an arbitrarily small vicinity of the contours $\partial\mathcal{D}_{R,L,s}$ and $\Sigma_{1,3,s} \cap \mathcal{D}_\infty$. Using (5.15) and (5.2) one gets the approximation to $Y(k)$ up to the order of $x^{-1-\epsilon}$, uniform outside this vicinity. This completes the solution to the RH problem (2.52) in the large x limit.

6. Time-dependent correlation functions: space-like region

In this section we derive the asymptotic expressions for the correlation functions $G_h(x,t)$ and $G_e(x,t)$ in the space-like region $|x| > 2|t|$. The asymptotic expressions are derived assuming that $x, t \to \infty$ at a fixed value of the parameter k_s defined in (5.1). The explicit expressions are written assuming $x > 0$ and $t > 0$; all other cases follow from the relations (2.1) and (2.2).

All the calculations will be similar to those for the $t = 0$ case. In section 6.1 we calculate the functions B_{ab} and C_{ab} (2.16) from the large k expansion (2.55) of $Y(k)$. Having these functions we calculate the Fredholm determinant $\det(\hat{I} + \hat{V})$ in the large x limit, see equation (6.8). The constant $C(\eta)$ entering (6.8) is calculated in section 6.2. This constant is found to be the same as in the $t = 0$ case. We give the answer for the correlation functions $G_h(x,t)$ and $G_e(x,t)$ in section 6.3.

6.1. Large x asymptotics of B_{ab}, C_{ab} and of the Fredholm determinant $\det(\hat{I} + \hat{V})$

In this subsection we use the results of section 5 on the asymptotic solution of the RH problem (2.52) to calculate the functions B_{ab}, C_{ab} and the Fredholm determinant $\det(\hat{I} + \hat{V})$ in the large x limit.

It follows from (5.68), (5.2), (5.15), (5.24), (5.56), (5.59) and (5.65) that in the vicinity of $k = \infty$ the following uniform approximation is valid:

$$Y = M_\infty N_\infty Q_\infty V + \mathcal{O}(x^{-1-\epsilon}). \tag{6.1}$$

Recall that the matrix M_∞ is given by equation (5.66), the matrix N_∞ by equation (5.60), the matrix Q_∞ by (3.21), the matrix V by (5.3). The approximation (6.1) being uniform, the functions B_{ab} and C_{ab} (2.16) can be calculated in the large x limit from the large k expansion (2.55) of equation (6.1). In particular,

$$B_{--} = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{e}^{-\frac{\mathrm{i}\eta}{2}} \operatorname{cosec}\frac{\eta}{2}\left[\left(\frac{k_s-1}{k_s+1}\right)^{-\frac{\pi-\eta}{2\pi}}\varkappa_L - \left(\frac{k_s-1}{k_s+1}\right)^{\frac{\pi+\eta}{2\pi}}\varkappa_R\right][1 + \mathcal{O}(x^{-\epsilon})]. \tag{6.2}$$

The function b_{++} entering (2.22) has the form

$$b_{++} = \left\{2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t}\,\mathrm{e}^{\frac{\mathrm{i}\eta}{2}} \sin\frac{\eta}{2}\left[\left(\frac{k_s-1}{k_s+1}\right)^{\frac{\pi-\eta}{2\pi}}\varkappa_L - \left(\frac{k_s-1}{k_s+1}\right)^{-\frac{\pi+\eta}{2\pi}}\varkappa_R\right]\right.$$
$$\left. - 2\sin^2\frac{\eta}{2}\left(\frac{k_s-1}{k_s+1}\right)^{\frac{\eta}{\pi}} G_0\right\}[1 + \mathcal{O}(x^{-\epsilon})]. \tag{6.3}$$

The expressions for the other potentials are quite bulky, so we write down the differential equations (2.41) and (2.42) in the large x limit omitting intermediate calculations

$$\partial_x \ln\det(\hat{I}+\hat{V}) = \frac{\mathrm{i}\eta}{\pi} - \frac{k_s^2}{k_s^2-1}\frac{\eta^2}{2\pi^2 x} + 2\mathrm{i}\kappa^{-1}\left(\varkappa_R^2 - \varkappa_L^2\right) - \frac{(k_s-1)\kappa+2}{\left(k_s^2-1\right)\kappa}\alpha_R - \frac{(k_s+1)\kappa-2}{\left(k_s^2-1\right)\kappa}\alpha_L + \mathcal{O}(x^{-1-\epsilon}) \tag{6.4}$$

and

$$\partial_t \ln\det(\hat{I}+\hat{V}) = \frac{2k_s}{k_s^2-1}\frac{\eta^2}{2\pi^2 x} + \alpha_R + \alpha_L + \mathcal{O}(x^{-1-\epsilon}) \tag{6.5}$$

where

$$\alpha_R = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}(1-\mathrm{e}^{-\mathrm{i}\eta})\varkappa_R G_0\kappa^{-1}\left(\frac{k_s-1}{k_s+1}\right)^{\frac{3(\pi+\eta)}{2\pi}} \tag{6.6}$$

$$\alpha_L = -2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}(1-\mathrm{e}^{-\mathrm{i}\eta})\varkappa_L G_0\kappa^{-1}\left(\frac{k_s-1}{k_s+1}\right)^{-\frac{3(\pi-\eta)}{2\pi}} \tag{6.7}$$

and κ is given by equation (3.64).

Integrating equations (6.4) and (6.5) asymptotically one gets

$$\ln\det(\hat{I}+\hat{V}) = \frac{\mathrm{i}\eta}{\pi}x - \left(\frac{\eta}{2\pi}\right)^2 \ln\left[x^2\left(1-k_s^{-2}\right)\right] + C(\eta) + \mathcal{O}(x^{-\epsilon}). \tag{6.8}$$

Note that $x^2\left(1-k_s^{-2}\right) = x_R x_L$. We calculate the integration constant $C(\eta)$ in section 6.2. It will be the same as for the $t=0$ case.

6.2. Calculation of $C(\eta)$

In this subsection we calculate the constant $C(\eta)$ in (6.8) using the differential equation (2.37). The calculation will resemble that of section 4.2.

Consider the first term on the right-hand side of equation (2.37). Split the integral in equation (2.38) into two parts and define

$$J_R = \frac{1}{1-\mathrm{e}^{-\mathrm{i}\eta}}\int_0^1 \mathrm{d}k[\vec{f}(k)]^T\sigma_2\partial_k\vec{f}(k) \tag{6.9}$$

$$J_L = \frac{1}{1-\mathrm{e}^{-\mathrm{i}\eta}}\int_{-1}^0 \mathrm{d}k[\vec{f}(k)]^T\sigma_2\partial_k\vec{f}(k). \tag{6.10}$$

Consider J_R first. The large x asymptotics of the function $\vec{f}(k)$ in equation (6.9) is calculated as follows. Use equation (2.53)

$$\vec{f}(k) = Y_+(k)\vec{e}(k) \qquad k\in\Sigma_2. \tag{6.11}$$

Substituting (5.2) and (5.3) into (6.11) and using (2.11) and (2.25) one gets

$$\vec{f}(k) = Z_+(k)\vec{e}_0(k) \qquad k\in\Sigma_2 \tag{6.12}$$

where

$$\vec{e}_0(k) = \begin{pmatrix} e_+^0(k) \\ e_-(k)\end{pmatrix} \qquad e_+^0(k) = \frac{\mathrm{i}(1-\mathrm{e}^{\mathrm{i}\eta})}{2\sqrt{\pi}}\,\mathrm{e}^{-\tau(k)/2} \tag{6.13}$$

the matrix $Z(k)$ solves the RH problem (5.10) and the functions $e_-(k)$ and $\tau(k)$ are defined by equation (2.11). It follows from (5.15) and (5.16) that

$$Z_+(k) = U_+(k)\mu_1(k) \qquad k \in \Sigma_2 \tag{6.14}$$

where $U(k)$ solves the RH problem (5.22). Substituting equation (6.14) into (6.11) and performing the analytic continuation of formula (5.68) in complete analogy with the analytic continuation of formula (3.67) discussed in section 4.2 and shown in figure 6 we obtain the approximation to $\vec{f}(k)$

$$\vec{f} = M_\infty N_R \vec{g}_R[I + \mathcal{O}(x^{-1-\epsilon})] \tag{6.15}$$

uniform for $k \in [0, 1]$. The matrices M_∞ and N_R are defined in section 5.5, the function $\vec{g}$ is given by the formula

$$\vec{g}_R(k) = \begin{pmatrix} g_+^R(k) \\ g_-^R(k) \end{pmatrix} = [Q_R(k)]_+ \mu_1(k)\vec{e}_0(k) \qquad k \in [0, 1]. \tag{6.16}$$

Substituting into (6.16) the expression for $Q_R(k)$, defined by equations (5.37), (5.33) and (3.31), and using the identity (4.13) one gets

$$\begin{aligned} g_+^R(k; x, t) &= -\frac{\sqrt{\pi}\, \mathrm{e}^{\frac{\mathrm{i}\eta}{4}}\, \mathrm{e}^{\tau(k)/2-\tau(1)}}{\Gamma\left(-\frac{\eta}{2\pi}\right)} \frac{{}_1F_1\left[\frac{\eta}{2\pi}+1, 1; \mathrm{i}x_R(\lambda_R - 1)\right]}{[x_R(k+1)]^{\frac{\eta}{2\pi}}} \left(\frac{k-1}{\lambda_R - 1}\right)^{\frac{\eta}{2\pi}} \\ g_-^R(k; x, t) &= \frac{\sqrt{\pi}\, \mathrm{e}^{-\frac{\mathrm{i}\eta}{4}}\, \mathrm{e}^{\tau(k)/2}}{\Gamma\left(\frac{\eta}{2\pi}\right)\sin\frac{\eta}{2}} \frac{{}_1F_1\left[\frac{\eta}{2\pi}, 1; \mathrm{i}x_R(\lambda_R - 1)\right]}{[x_R(k+1)]^{-\frac{\eta}{2\pi}}} \left(\frac{k-1}{\lambda_R - 1}\right)^{-\frac{\eta}{2\pi}} \end{aligned} \tag{6.17}$$

where $\lambda_R = \lambda_R(k)$ is defined by equation (5.32).

Using the uniform estimate (6.15) and equations (6.17), (5.60)–(5.67) one can derive the following estimate for (6.9):

$$J_R = \frac{1}{1-\mathrm{e}^{-\mathrm{i}\eta}} \int_0^1 \mathrm{d}k [\vec{g}_R(k)]^T \sigma_2 \partial_k \vec{g}_R(k)[1 + \mathcal{O}(x^{-1-\epsilon})]. \tag{6.18}$$

Performing the integral in (6.18) asymptotically one gets

$$J_R = \frac{\mathrm{i}x_R}{2\pi} - \frac{\eta}{2\pi^2}\ln x_R + \frac{1}{2}C_1(\eta) + \mathcal{O}(x^{-\epsilon}) \tag{6.19}$$

where $C_1(\eta)$ is defined by equation (4.17).

The large x asymptotics of J_L is calculated similarly, yielding

$$J_L = \frac{\mathrm{i}x_L}{2\pi} - \frac{\eta}{2\pi^2}\ln x_L + \frac{1}{2}C_1(\eta) + \mathcal{O}(x^{-\epsilon}). \tag{6.20}$$

We have calculated the large x asymptotics of the first term on the right-hand side of equation (2.37). By analysis of equation (2.40) one can show that the second term on the right-hand side of equation (2.37) is of the order of $x^{-\epsilon}$. Therefore, the large x asymptotics of (2.37) is

$$\begin{aligned} \partial_\eta \ln\det(\hat{I} + \hat{V}) &= J_R + J_L + \mathcal{O}(x^{-\epsilon}) \\ &= \frac{\mathrm{i}x}{\pi} - \frac{\eta}{2\pi^2}\ln(x_R x_L) + C_1(\eta) + \mathcal{O}(x^{-\epsilon}). \end{aligned} \tag{6.21}$$

Integrating equation (6.21) over η, taking into account the initial condition (2.43) and comparing the resulting expression with (6.8) one finds that $C(\eta)$ is given by the same expression (4.19) as in the $t = 0$ case.

6.3. Time-dependent correlation functions in the space-like region: the results

In this subsection we write down the asymptotic expressions for the correlation functions $G_h(x,t)$ and $G_e(x,t)$ in the space-like region $|x| > 2|t|$. The asymptotic expressions are given assuming that $x, t \to \infty$ at a fixed value of the parameter k_s defined in (5.1). The explicit expressions are written assuming $x > 0$ and $t > 0$; all other cases follow from relations (2.1) and (2.2).

The correlation function $G_h(x,t)$ is given by formula (2.32). The asymptotics of B_{--} is given by (6.2); the asymptotics of $\det(\hat{I} + \hat{V})$ by (6.8) with the constant $C(\eta)$ given by (4.19). Recall that $z = \exp(\mathrm{i}\eta)$, therefore the point $z = 1/2$ corresponds to $\eta = \mathrm{i}\ln 2$. The leading contribution to the asymptotics of $G_h(x,t)$ comes from the first term on the right-hand side of equation (2.32), yielding

$$\begin{aligned} G_h(x,t) = {} & \frac{\Xi}{2\mathrm{i}} \frac{\mathrm{e}^{-x\ln 2/\pi}}{(x+2t)^{2\Delta}(x-2t)^{2\bar{\Delta}}} \exp\left[\mathrm{i}(x-x_0) - \frac{\mathrm{i}\ln 2}{\pi}\ln(x+2t)\right] \\ & - \frac{\Xi}{2\mathrm{i}} \frac{\mathrm{e}^{-x\ln 2/\pi}}{(x-2t)^{2\Delta}(x+2t)^{2\bar{\Delta}}} \exp\left[-\mathrm{i}(x-x_0) + \frac{\mathrm{i}\ln 2}{\pi}\ln(x-2t)\right] \end{aligned} \tag{6.22}$$

where the normalization constant Ξ is defined by equation (4.23) and the phase shift x_0 by equation (4.22). The anomalous exponents are given by

$$\Delta = \frac{1}{2} - \frac{1}{8}\left(\frac{\ln 2}{\pi}\right)^2 \qquad \bar{\Delta} = -\frac{1}{8}\left(\frac{\ln 2}{\pi}\right)^2. \tag{6.23}$$

The relative correction to (6.22) is of the order of x^{-1}.

Show that the second term on the right-hand side of equation (2.32) does not contribute to the asymptotics (6.22). By virtue of (6.8), the determinant $\det(\hat{I} + \hat{V})$ is of the order of $\exp(x\ln r/\pi)$ on the integration contour γ, where γ is the circle $|z| = r$, figure 1. Therefore, for $r < 1/2$ and for sufficiently large x the contribution of the second term is negligible.

The correlation function $G_e(x,t)$ is calculated similarly, yielding

$$G_e(x,t) = -G_h^*(x,t) + \mathrm{e}^{\mathrm{i}t} G_0(x,t) \frac{\mathrm{e}^{-x\ln 2/\pi}\,\mathrm{e}^{C(\mathrm{i}\ln 2)}}{(x^2-4t^2)^{2\bar{\Delta}}} \left(\frac{x-2t}{x+2t}\right)^{\mathrm{i}\ln 2/\pi} \tag{6.24}$$

where $G_h(x,t)$ is given by equation (6.22), the function $G_0(x,t)$ is defined by equation (2.19) and $\bar{\Delta}$ defined by (6.23).

Compare the result (6.24) with the general relation (2.3). One can see that the last term on the right-hand side of equation (6.24) corresponds to the averaged anticommutator of fermion fields in equation (2.3). It contains a rapidly oscillating factor $\exp(\mathrm{i}t)$. This means that although formally present in the long distance expansion of the correlation function G_e this term is irrelevant at energies smaller than the chemical potential.

7. Riemann–Hilbert problem at $t \neq 0$. Time-like region

In this section we find the asymptotic solution of the Riemann–Hilbert problem (2.52) for $|x| < 2|t|$. We assume that $x > 0$ and $t > 0$. We will consider the large t asymptotics at a fixed value of the parameter k_s defined in (5.1).

In section 7.1 we reformulate the RH problem (2.52) so as to make it similar to the RH problem (3.3). In section 7.2 we repeat the construction made for the $t = 0$ case in sections 3.2 and 3.3. In section 7.3 we give an explicit solution to the jump relation (7.7b) in a vicinity of the points $k = \pm 1$ and $k = k_s$. In section 7.4 we perform the matching procedure analogous to that of section 3.5.

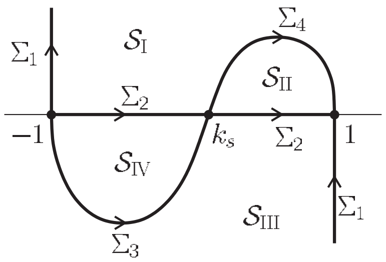

Figure 11. The conjugation contour $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$ for the RH problem (7.7). The contour Σ_1 consists of two semi-infinite lines parallel to the imaginary axis.

7.1. Reformulation of the Riemann–Hilbert problem (2.52)

In this subsection we reformulate the RH problem (2.52) so as to make use of the results of sections 5.3 and 5.4.

Consider a contour shown in figure 11. This contour divides the complex k-plane into four regions, denoted as $\mathcal{S}_{\rm I}$ through $\mathcal{S}_{\rm IV}$. Represent the solution $Y(k)$ of the RH problem (2.52) as a product of two matrices

$$Y(k) = Z(k)V(k) \tag{7.1}$$

where

$$V(k) = \begin{pmatrix} 1 & 0 \\ 2\mathrm{i}\,\mathrm{e}^{\tau(k)}\varrho_1(k) & 1 \end{pmatrix} \begin{pmatrix} 1 & -2\sin^2\frac{\eta}{2}[E(k) - \mathrm{i}\,\mathrm{e}^{-\tau(k)}\varrho_2(k)] \\ 0 & 1 \end{pmatrix}. \tag{7.2}$$

The piecewise-constant functions ϱ_1 and ϱ_2 are defined by

$$\varrho_1(k) = \begin{cases} \mathrm{e}^{\mathrm{i}\eta} & k \in \mathcal{S}_{\rm II} \\ -\mathrm{e}^{-\mathrm{i}\eta} & k \in \mathcal{S}_{\rm IV} \\ 0 & \text{otherwise} \end{cases} \tag{7.3}$$

and

$$\varrho_2(k) = \begin{cases} 1 & k \in \mathcal{S}_{\rm III} \quad \mathrm{Im}\, k > 0 \\ -1 & k \in \mathcal{S}_{\rm I} \quad \mathrm{Im}\, k < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

The contour Σ_1 is chosen so that the real part of $\tau(k)$ is positive for all $k \in \Sigma_1$ except $k = \pm 1$ and

$$\mathrm{Re}\,\tau(k) \to +\infty \qquad \text{as} \quad k \to \infty \quad k \in \Sigma_1 \tag{7.5}$$

Note that condition (7.5) implies

$$V(k) \to I \quad \text{as} \quad k \to \infty. \tag{7.6}$$

Using equations (2.52), (7.1) and (7.2) and the property (7.6) one gets the following RH problem for $Z(k)$

$$\begin{aligned} &\text{(a) } Z(k) \text{ is analytic in } \mathbb{C}\backslash\Sigma \\ &\text{(b) } Z_+(k) = Z_-(k)\mu_i(k) \qquad k \in \Sigma_i \quad i = 1, 2, 3, 4 \\ &\text{(c) } Z(k) \to I \quad \text{as} \quad k \to \infty \end{aligned} \tag{7.7}$$

where the jump matrices μ_1, μ_2 and μ_3 are given by equations (5.17), (5.20) and (5.18), respectively. The matrix μ_4 is defined as

$$\mu_4 = \begin{pmatrix} 1 & 0 \\ 2\mathrm{i}\,\mathrm{e}^{\tau(k)}\,\mathrm{e}^{\mathrm{i}\eta} & 1 \end{pmatrix}. \tag{7.8}$$

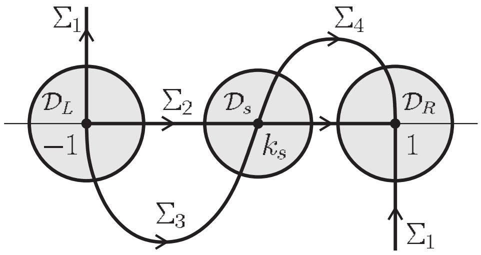

Figure 12. The contour used in the factorization of the RH problem (7.7). The discs $\mathcal{D}_R$, $\mathcal{D}_L$ and $\mathcal{D}_s$ are shaded in grey.

7.2. Factorization of the Riemann–Hilbert problem (7.7)

In this subsection we sketch the scheme of the analysis of the RH problem (7.7) in the large t limit. This scheme is similar to that given in section 5.3.

The matrices μ_1, μ_3 and μ_4 converge to the identity matrix, as $t \to \infty$, everywhere on their conjugation contours except at the points $k = \pm 1$ and $k = k_s$. Introduce three discs $\mathcal{D}_L$, $\mathcal{D}_R$ and $\mathcal{D}_s$ as shown in figure 12. Denote the domain $\mathbb{C}\backslash(\mathcal{D}_L \cup \mathcal{D}_R \cup \mathcal{D}_s)$ as D_∞. Represent the solution $Z(k)$ of the RH problem (7.7) as a product of three matrices

$$Z = KSQ. \tag{7.9}$$

Here the matrix $Q(k)$ is defined by

$$Q(k) = Q_\alpha(k) \qquad k \in \mathcal{D}_\alpha \quad \alpha = L, R, s, \infty. \tag{7.10}$$

The matrix Q_∞ is given by equation (3.18). The matrices Q_R and Q_L are defined in $\mathcal{D}_R$ and $\mathcal{D}_L$, respectively, and satisfy there the jump relations

$$[Q_R(k)]_+ = [Q_R(k)]_- \mu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_R \quad i = 1, 2, 4 \tag{7.11}$$

and

$$[Q_L(k)]_+ = [Q_L(k)]_- \mu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_L \quad i = 1, 2, 3. \tag{7.12}$$

The matrix $Q_s(k)$ is defined in $\mathcal{D}_s$ and satisfies

$$[Q_s(k)]_+ = [Q_s(k)]_- \mu_i(k) \qquad k \in \Sigma_i \cap \mathcal{D}_s \quad i = 2, 3, 4. \tag{7.13}$$

We give solutions to the jump relations (7.11), (7.12) and (7.13) in section 7.3.

The matrix $S(k)$ in the decomposition (7.9) is the solution to the RH problem (5.29) with the conjugation matrices $\theta_{R,L,s}$ defined by equation (5.30). The matrices Q_R, Q_L and Q_s in (5.30) solve the jump relations (7.11), (7.12) and (7.13), respectively.

Like the spacelike case, the matrix K can be replaced by the identity matrix.

7.3. Exact solutions to the jump relations (7.11), (7.12) and (7.13)

In this subsection we solve explicitly the jump relations (7.11), (7.12) and (7.13). We use the results of section 5.4.

The jump relation (7.12) coincides with the jump relation (5.28). Therefore one can use the solution (5.39) to this relation. To obtain a solution to the jump relation (7.11) consider the conjugation matrices μ_1, μ_2 and μ_4 defined on the contours Σ_1, Σ_2 and Σ_4, shown in figure 12. These matrices satisfy the following involutions:

$$\begin{aligned} \mu_1(k; x, t, \eta) &= P_1^{-1} \mu_3(k; -x, -t, -\eta) P_1 \\ \mu_2(k; \eta) &= P_1^{-1} \mu_2(k; -\eta) P_1 \\ \mu_4(k; x, t, \eta) &= P_1^{-1} \mu_1(k; -x, -t, -\eta) P_1 \end{aligned} \tag{7.14}$$

where P_1 is a k-independent matrix defined by

$$P_1 = \begin{pmatrix} 0 & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\eta/2}\sin\frac{\eta}{2} \\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta/2}\operatorname{cosec}\frac{\eta}{2} & 0 \end{pmatrix}. \tag{7.15}$$

Denote the solution (5.37) to the jump relation (5.27) by $Q_R^{\mathrm{sl}}(k; x, t, \eta)$. Comparing the conjugation contours in the discs $\mathcal{D}_R$ shown in figures 9 and 12 and using the involutions (7.14) a solution Q_R to the jump relation (7.11) can be found from Q_R^{sl}

$$Q_R(k; x, t, \eta) = P_1 Q_R^{\mathrm{sl}}(k; -x, -t, -\eta) P_1^{-1}. \tag{7.16}$$

Finally, write down a solution to the jump relation (7.13):

$$Q_s(k) = \begin{pmatrix} 1 & 0 \\ q(k) & 1 \end{pmatrix} Q_\infty(k) \tag{7.17}$$

where

$$q(k) = \int_{\Sigma_s} \frac{\mathrm{d}p}{2\pi\mathrm{i}} \left(\frac{1-p}{1+p}\right)^{-\frac{\eta}{\pi}} \frac{2\mathrm{i}\,\mathrm{e}^{\tau(p)}}{p-k} \tag{7.18}$$

and the integral runs along the contour Σ_s which consists of the part of the contour Σ_3 lying outside $\mathcal{D}_L$ and the part of the contour Σ_4 lying outside $\mathcal{D}_R$, see figure 12.

7.4. Solution to the Riemann–Hilbert problem (5.29) in the large t limit

In this subsection we solve the RH problem (5.29) in the large t limit. The procedure will resemble that of section 5.5.

The conjugation matrix θ_L is approximated by a matrix $\tilde{\theta}_L$

$$\theta_L = \tilde{\theta}_L + \mathcal{O}(t^{-1-\epsilon}) \tag{7.19}$$

uniformly for $k \in \partial\mathcal{D}_L$, where $\tilde{\theta}_L$ is defined by equations (5.46) and (5.48). The conjugation matrix θ_R is approximated by a matrix $\tilde{\theta}_R$

$$\theta_R = \tilde{\theta}_R + \mathcal{O}(t^{-1-\epsilon}) \tag{7.20}$$

uniformly for $k \in \partial\mathcal{D}_R$, where the matrix $\tilde{\theta}_R$ is given by equation (5.42) and υ_R is defined as

$$\upsilon_R(k; x, t, \eta) = -P_1 \upsilon_R^{\mathrm{sl}}(k; -x, -t, -\eta) P_1. \tag{7.21}$$

Here we use the notation $\upsilon_R^{\mathrm{sl}}(k; x, t, \eta)$ for the matrix given by equation (5.43). The matrix P_1 is defined by equation (7.15). The conjugation matrix θ_s is approximated by a matrix $\tilde{\theta}_s$

$$\theta_s = \tilde{\theta}_s[I + \mathcal{O}(t^{-3/2})] \tag{7.22}$$

uniformly for $k \in \partial\mathcal{D}_s$. The matrix $\tilde{\theta}_s$ is given by equation (5.52) where $\upsilon_s(k)$ is calculated using (5.30), (7.17) and (7.18)

$$\upsilon_s = \begin{pmatrix} 0 & 0 \\ -2\left(\frac{1-k_s}{1+k_s}\right)^{-\frac{\eta}{\pi}} G_0^* & 0 \end{pmatrix}. \tag{7.23}$$

The asymptotic estimate for the matrix S is given by equation (5.55) where the matrix $\tilde{S}$ is the solution of the RH problem (5.54) with the matrices $\tilde{\theta}_{R,L,s}$ defined in the paragraph above. Like the space-like case, solve the RH problem (5.54) using the decomposition (5.56). The matrix N_∞ entering the definition (5.59) is given by equation (5.60) with matrices A_R and A_L given by

$$A_R = \frac{2}{\kappa} \begin{pmatrix} \varkappa_L^2 + \frac{\mathrm{i}}{4t}\left(\frac{\eta}{2\pi}\right)^2 \frac{1}{1-k_s} & -\varkappa_R \sin\frac{\eta}{2}\,\mathrm{e}^{-\mathrm{i}t}\left(\frac{1+k_s}{1-k_s}\right)^{\frac{\pi+\eta}{2\pi}} \\ -\varkappa_L \operatorname{cosec}\frac{\eta}{2}\,\mathrm{e}^{\mathrm{i}t}\left(\frac{1+k_s}{1-k_s}\right)^{\frac{\pi-\eta}{2\pi}} & \varkappa_R^2 - \frac{\mathrm{i}}{4t}\left(\frac{\eta}{2\pi}\right)^2 \frac{1}{1-k_s} \end{pmatrix} + \mathcal{O}(t^{-1-\epsilon}) \tag{7.24}$$

and

$$A_L = \frac{2}{\kappa}\begin{pmatrix} -\varkappa_R^2 - \frac{\mathrm{i}}{4t}\left(\frac{\eta}{2\pi}\right)^2 \frac{1}{1+k_s} & -\varkappa_L \sin\frac{\eta}{2}\,\mathrm{e}^{-\mathrm{i}t}\left(\frac{1-k_s}{1+k_s}\right)^{\frac{\pi-\eta}{2\pi}} \\ -\varkappa_R \operatorname{cosec}\frac{\eta}{2}\,\mathrm{e}^{\mathrm{i}t}\left(\frac{1-k_s}{1+k_s}\right)^{\frac{\pi+\eta}{2\pi}} & -\varkappa_L^2 + \frac{\mathrm{i}}{4t}\left(\frac{\eta}{2\pi}\right)^2 \frac{1}{1+k_s} \end{pmatrix} + \mathcal{O}(t^{-1-\epsilon}). \tag{7.25}$$

Here

$$\begin{aligned} \varkappa_R &= -\frac{\mathrm{i}\pi\,\mathrm{e}^{\mathrm{i}x}(4t)^{-1-\frac{\eta}{\pi}}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)\mathrm{e}^{\frac{\mathrm{i}\eta}{2}}\sin\frac{\eta}{2}}\left(\frac{1}{1-k_s^2}\right)^{\frac{\pi+\eta}{2\pi}} \\ \varkappa_L &= -\frac{\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}x}(4t)^{-1+\frac{\eta}{\pi}}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\mathrm{e}^{-\frac{\mathrm{i}\eta}{2}}\sin\frac{\eta}{2}}\left(\frac{1}{1-k_s^2}\right)^{\frac{\pi-\eta}{2\pi}} \end{aligned} \tag{7.26}$$

and κ is defined by equation (3.64). The matrix $M_\infty(k)$ in the decomposition (5.56) is given by equation (5.66) with A_s defined by equation (5.67).

8. Time-dependent correlation functions: time-like region, $k_s \neq 0$

In this section we derive the asymptotics of the correlation functions $G_e(x,t)$ and $G_h(x,t)$ in the time-like region $|x| < 2|t|$. The asymptotic expressions will be given assuming that $x, t \to \infty$ at a fixed value of the parameter k_s defined in equation (5.1). It will also be assumed that $x > 0$ and $t > 0$; all other cases follow from relations (2.1) and (2.2).

The calculations in this section are similar to those given in sections 4 and 6. We will therefore omit most technical details already explained in those sections and focus on the particulars of the time-like case. In section 8.1 we give the large t asymptotics of B_{--}, b_{++} and $\det(\hat{I}+\hat{V})$. In section 8.2 we perform the contour integral entering representations (2.21) and (2.22) and obtain the asymptotic expressions for the correlation functions $G_e(x,t)$ and $G_h(x,t)$.

8.1. Large t asymptotics of B_{ab}, C_{ab} and of the Fredholm determinant $\det(\hat{I}+\hat{V})$

The asymptotic expressions for the functions B_{--} and b_{++} are calculated similarly as in section 6.1, yielding

$$\begin{aligned} B_{--} = \Bigg\{ 2\left(\frac{1+k_s}{1-k_s}\right)^{\frac{\eta}{\pi}} G_0^* + 2\,\mathrm{e}^{\mathrm{i}t}\operatorname{cosec}\frac{\eta}{2}\Bigg[\left(\frac{1-k_s}{1+k_s}\right)^{-\frac{\pi-\eta}{2\pi}}\varkappa_L \\ + \left(\frac{1-k_s}{1+k_s}\right)^{\frac{\pi+\eta}{2\pi}}\varkappa_R\Bigg]\Bigg\}[1+\mathcal{O}(t^{-\epsilon})] \end{aligned} \tag{8.1}$$

and

$$b_{++} = 2\,\mathrm{e}^{-\mathrm{i}t}\sin\frac{\eta}{2}\left[\left(\frac{1-k_s}{1+k_s}\right)^{\frac{\pi-\eta}{2\pi}}\varkappa_L + \left(\frac{1-k_s}{1+k_s}\right)^{-\frac{\pi+\eta}{2\pi}}\varkappa_R\right][1+\mathcal{O}(t^{-\epsilon})] \tag{8.2}$$

where $\varkappa_R$ and $\varkappa_L$ are given by equation (7.26). The large t asymptotics of the Fredholm determinant $\det(\hat{I}+\hat{V})$ is

$$\ln\det(\hat{I}+\hat{V}) = \frac{\mathrm{i}\eta}{\pi}x - \left(\frac{\eta}{2\pi}\right)^2 \ln\left[4t^2\left(1-k_s^2\right)\right] + \tilde{C}(\eta) + \mathcal{O}(t^{-\epsilon}). \tag{8.3}$$

The constant $\tilde{C}(\eta)$ can, in principle, be calculated in the same manner as the constant $C(\eta)$ for the space-like region. There is, however, a difference which did not allow us to obtain a closed analytical expression for $\tilde{C}(\eta)$ so far. Unlike the space-like case, the second term on the right-hand side of equation (2.37) is not small as a function of the asymptotic parameter t. Therefore, the calculation of $\tilde{C}(\eta)$ requires the asymptotic analysis of expression (2.40), which is straightforward but cumbersome. This is in contrast to the high temperature case, where the calculation of such a constant was done using an additional differential equation involving the functions B_{ab} and C_{ab} only [11]. Such an equation does not exist in the zero temperature case, considered here.

8.2. Time-dependent correlation functions: time-like region, $k_s \neq 0$. The results

In this subsection we write down the asymptotic expressions for the correlation functions $G_h(x,t)$ and $G_e(x,t)$ in the time-like region $|x| < 2|t|$. The asymptotic expressions are given assuming that $x, t \to \infty$ at a fixed value of the parameter k_s defined in (5.1). The explicit expressions are written assuming $x > 0$ and $t > 0$; all other cases follow from relations (2.1) and (2.2).

Consider the representations (2.32) and (2.33). Applying the same arguments as given in section 6.3 one can show that the contributions from the integrals along the contour γ in (2.32) and (2.33) can be neglected in the large t limit. Substituting the results (8.1), (8.2) and (8.3) in equations (2.32) and (2.33) we find

$$\begin{aligned} G_h(x,t) = {} & \frac{\tilde{\Xi}}{2\mathrm{i}} \frac{\mathrm{e}^{-x\ln 2/\pi}}{(2t+x)^{2\Delta}(2t-x)^{2\bar{\Delta}}} \exp\left[\mathrm{i}(x-x_0) - \frac{\mathrm{i}\ln 2}{\pi}\ln(2t+x)\right] \\ & + \frac{\tilde{\Xi}}{4\mathrm{i}} \frac{\mathrm{e}^{-x\ln 2/\pi}}{(2t-x)^{2\Delta}(2t+x)^{2\bar{\Delta}}} \exp\left[-\mathrm{i}(x-x_0) + \frac{\mathrm{i}\ln 2}{\pi}\ln(2t-x)\right] \\ & + \frac{\mathrm{e}^{-\mathrm{i}t}}{2} G_0^*(x,t) \frac{\mathrm{e}^{-x\ln 2/\pi}\,\mathrm{e}^{C(\mathrm{i}\ln 2)}}{(4t^2-x^2)^{2\bar{\Delta}}} \left(\frac{2t-x}{2t+x}\right)^{-\mathrm{i}\ln 2/\pi} \end{aligned} \qquad (8.4)$$

and

$$\begin{aligned} G_e(x,t) = {} & -\frac{\tilde{\Xi}}{\mathrm{i}} \frac{\mathrm{e}^{-x\ln 2/\pi}}{(2t-x)^{2\Delta}(2t+x)^{2\bar{\Delta}}} \exp\left[\mathrm{i}(x-x_0) - \frac{\mathrm{i}\ln 2}{\pi}\ln(2t-x)\right] \\ & - \frac{\tilde{\Xi}}{2\mathrm{i}} \frac{\mathrm{e}^{-x\ln 2/\pi}}{(2t+x)^{2\Delta}(2t-x)^{2\bar{\Delta}}} \exp\left[-\mathrm{i}(x-x_0) + \frac{\mathrm{i}\ln 2}{\pi}\ln(2t+x)\right] \end{aligned} \qquad (8.5)$$

where dimensions Δ and $\bar{\Delta}$ are given by equation (6.23), the phase shift x_0 by equation (4.22). The normalization constant $\tilde{\Xi}$ can be obtained from the constant Ξ given in equation (4.23) by replacing the constant $C(\eta)$ with $\tilde{C}(\eta)$.

9. Time dependent correlation functions: time-like region, $x = 0$

In this section we consider the special case of $x = 0$. The large t asymptotics of the correlation functions $G_e(0,t)$ and $G_h(0,t)$ are given assuming that $t > 0$; the case of negative t follows from relation (2.2). In the derivation of the asymptotical expressions we use the results of section 8.

For $x = 0$ it is convenient to calculate the large t asymptotics of the integrals (2.6) and (2.7) without the deformation of the integration contour leading to (2.32) and (2.33). Substituting asymptotic formulae (8.2) and (8.3) in equation (2.7) we arrive at the expression

$$G_e(0,t) = \int_{-\pi}^{\pi} \mathrm{d}\eta \, \frac{F(\eta)}{1-\cos\eta} \frac{\mathrm{i}4^{-1+\frac{\eta}{\pi}-(\frac{\eta}{2\pi})^2} \mathrm{e}^{\tilde{C}(\eta)}}{\Gamma^2\left(\frac{\eta}{2\pi}\right)\mathrm{e}^{-\mathrm{i}\frac{\eta}{2}}} \exp\left\{\left[-1+\frac{\eta}{\pi}-2\left(\frac{\eta}{2\pi}\right)^2\right]\ln t\right\}$$
$$+ \int_{-\pi}^{\pi} \mathrm{d}\eta \, \frac{F(\eta)}{1-\cos\eta} \frac{\mathrm{i}4^{-1-\frac{\eta}{\pi}-(\frac{\eta}{2\pi})^2} \mathrm{e}^{\tilde{C}(\eta)}}{\Gamma^2\left(-\frac{\eta}{2\pi}\right)\mathrm{e}^{\mathrm{i}\frac{\eta}{2}}} \exp\left\{\left[-1-\frac{\eta}{\pi}-2\left(\frac{\eta}{2\pi}\right)^2\right]\ln t\right\}. \tag{9.1}$$

For large t the integrals on the right hand side of (9.1) can be evaluated in the saddle point approximation with $\ln t$ playing the role of the asymptotic parameter. The relative correction to the integrands in equation (9.1) is of the order of $t^{-\epsilon}$. At the saddle points, $\eta = \pm\pi$, one has $\epsilon = 0$ leading to an unknown constant factor. This factor can, in principle, be calculated using the approach presented in this paper. The resulting expression for G_e reads

$$G_e(0,t) = \frac{\text{const}}{\sqrt{t\ln t}}\left[1+\mathcal{O}\left(\frac{1}{\ln t}\right)\right]. \tag{9.2}$$

The integral (2.6) is evaluated similarly, yielding

$$G_h(0,t) = \frac{\text{const}'}{\sqrt{t\ln t}} + \frac{\text{const}''}{\sqrt{\ln t}}\,\mathrm{e}^{-\mathrm{i}t}G^*(0,t) \tag{9.3}$$

with the relative correction of the order of $1/\ln t$.

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